4. Central Forces

In this section we will study the three-dimensional motion of a particle in a central force potential. Such a system obeys the equation of motion

$$m\ddot{\mathbf{x}} = \mathrm{r}V(r) \tag{4.1}$$

where the potential depends only on $r = |\mathbf{x}|$. Since both gravitational and electrostatic forces are of this form, solutions to this equation contain some of the most important results in classical physics.

Our first line of attack in solving (4.1) is to use angular momentum. Recall that this is defined as

$$\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}}$$

We already saw in Section 2.2.2 that angular momentum is conserved in a central potential. The proof is straightforward:

$$\frac{d\mathbf{L}}{dt} = m\mathbf{x} \times \ddot{\mathbf{x}} = \mathbf{x} \times \mathrm{r}V = 0$$

where the final equality follows because $\mathrm{r}V$ is parallel to $\mathbf{x}$.

The conservation of angular momentum has an important consequence: all motion takes place in a plane. This follows because $\mathbf{L}$ is a fixed, unchanging vector which, by construction, obeys

$$\mathbf{L} \cdot \mathbf{x} = 0$$

So the position of the particle always lies in a plane perpendicular to $\mathbf{L}$. By the same argument, $\mathbf{L} \cdot \dot{\mathbf{x}} = 0$ so the velocity of the particle also lies in the same plane. In this way the three-dimensional dynamics is reduced to dynamics on a plane.

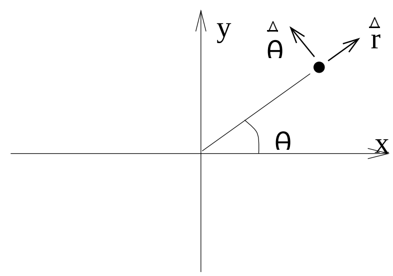
4.1 Polar Coordinates in the Plane

We've learned that the motion lies in a plane. It will turn out to be much easier if we work with polar coordinates on the plane rather than Cartesian coordinates. For this reason, we take a brief detour to explain some relevant aspects of polar coordinates.

To start, we rotate our coordinate system so that the angular momentum points in the z-direction and all motion takes place in the (x,y) plane. We then define the usual polar coordinates

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

Our goal is to express both the velocity and acceleration in polar coordinates. We introduce two unit vectors, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in the direction of increasing r and θ respectively as shown in the diagram. Written in Cartesian form, these vectors are



$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad , \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

Figure 11:

These vectors form an orthornormal basis at every point on the plane. But the basis itself depends on which angle θ we sit at. Moving in the radial direction doesn't change the basis, but moving in the angular direction we have

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \hat{\boldsymbol{\theta}} \quad , \quad \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \hat{\mathbf{r}}$$

This means that if the particle moves in a way such that θ changes with time, then the basis vectors themselves will also change with time. Let's see what this means for the velocity expressed in these polar coordinates. The position of a particle is written as the simple, if somewhat ugly, equation

$$\mathbf{x} = r\hat{\mathbf{r}}$$

From this we can compute the velocity, remembering that both r and the basis vector $\hat{\mathbf{r}}$ can change with time. We get

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \end{aligned} \tag{4.2}$$

The second term in the above expression arises because the basis vectors change with time and is proportional to the *angular velocity*, $\dot{\theta}$. (Strictly speaking, this is the angular speed. In the next section, we will introduce a vector quantity which is the angular velocity).

Di↵erentiating once more gives us the expression for acceleration in polar coordinates,

$$\ddot{\mathbf{x}} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\dot{\theta}$$

$$= (\ddot{r} \quad r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \qquad (4.3)$$

The two expressions (4.2) and (4.3) will be important in what follows.

An Example: Circular Motion

Let's look at an example that we're already all familiar with. A particle moving in a circle has $\dot{r} = 0$. If the particle travels with constant angular velocity $\dot{\theta} = \omega$ then the velocity in the plane is

$$\dot{\mathbf{x}} = r\omega\hat{\boldsymbol{\theta}}$$

so the speed in the plane in $v = |\dot{\mathbf{x}}| = r\omega$. Similarly, the acceleration in the plane is

$$\ddot{\mathbf{x}} = \quad r\omega^2\hat{\mathbf{r}}$$

The magnitude of the acceleration islaw, if we want a particle to travel in a circle, we need to supply a force$a = |\ddot{\mathbf{x}}| = r\omega^2 = v^2/r$. From Newton's second$F = mv^2/r$

towards the origin. This is known as a *centripetal force*.

4.2 Back to Central Forces

We've already seen that the three-dimensional motion in a central force potential actually takes place in a plane. Let's write the equation of motion (4.1) using the plane polar coordinates that we've just introduced. Since $V = V(r)$, the force itself can be written using

$$\nabla V = \frac{dV}{dr}\hat{\mathbf{r}}$$

and, from (4.3) the equation of motion becomes

$$m(\ddot{r} \quad r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = \quad \frac{dV}{dr}\hat{\mathbf{r}} \qquad (4.4)$$

The ✓ˆ component of this is particularly simple. It is

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

It looks as if we've found a new conserved quantity since we've learnt that

$l = r^2$✓˙ (4.5)

does not change with time. However, we shouldn't get too excited. This is something that we already know. To see this, let's look again at the angular momentum **L**. We already used the fact that the direction of **L** is conserved when restricting motion to the plane. But what about the magnitude of **L**? Using (4.2), we write

L = *m***x** ⇥ **x**˙ = *mr*ˆ**r** ⇥⇣*r*˙ˆ**r** + *r*✓˙ ✓ˆ⌘ = *mr*²✓˙ ⇣ˆ**r** ⇥✓ˆ⌘

Since ˆ**r** and ✓ˆ are orthogonal, unit vectors, ˆ**r**⇥✓ˆ is also a unit vector. The magnitude of the angular momentum vector is therefore

|**L**| = *ml*

and *l*, given in (4.5), is identified as the angular momentum per unit mass, although we will often be lazy and refer to *l* simply as the angular momentum.

Let's now look at the ˆ**r** component of the equation of motion (4.4). It is

$$m(\ddot{r} \quad r\dot{\theta}^2) = \quad \frac{dV}{dr}$$

Using the fact that $l = r^2$✓˙ is conserved, we can write this as

$$m\ddot{r} = \quad \frac{dV}{dr} + \frac{ml^2}{r^3} \tag{4.6}$$

It's worth pausing to reflect on what's happened here. We started in (4.1) with a complicated, three dimensional problem. We used the direction of the angular momentum to reduce it to a two dimensional problem, and the magnitude of the angular momentum to reduce it to a one dimensional problem. This was all possible because angular momentum is conserved.

This should give you some idea of how important conserved quantities are when it comes to solving anything. Roughly speaking, this is also why it's not usually possible to solve the *N*-body problem with *N* 3. In Section 5.1.5, we'll see that for the *N* = 2 mutually interacting particles, we can use the symmetry of translational invariance to solve the problem. But for N 3, we don't have any more conserved quantities to come to our rescue.

Returning to our main storyline, we can write (4.6) in the suggestive form

$$m\ddot{r} = \frac{dV_{\text{e↵}}}{dr} \tag{4.7}$$

where $V_{\text{e↵}}(r)$ is called the *e↵ective potential* and is given by

$$V_{\text{eff}}(r) = V(r) + \frac{ml^2}{2r^2} \tag{4.8}$$

The extra term, $ml^2/2r^2$ is called the *angular momentum barrier* (also known as the centrifugal barrier). It stops the particle getting too close to the origin, since there is must pay a heavy price in"e↵ective energy".

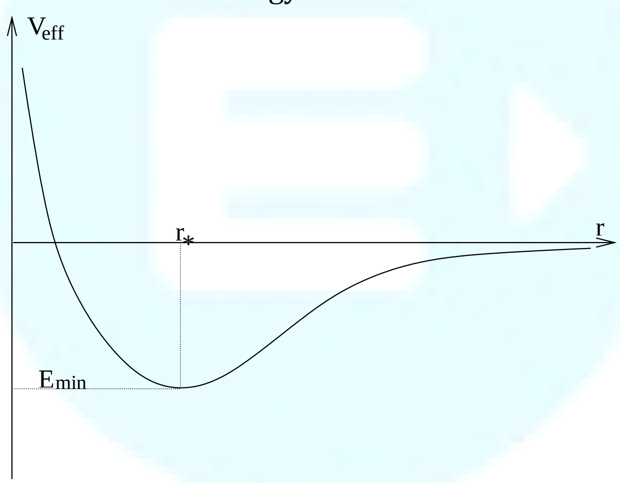


Figure 12: The e↵ective potential arising from the inverse square force law.

4.2.1 The E↵ective Potential: Getting a Feel for Orbits

Let's just check that the e↵ective potential can indeed be thought of as part of the energy of the full system. Using (4.2), we can write the energy of the full three dimensional problem as

$$
\begin{aligned}
E &= \frac{1}{2}m\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)
\end{aligned}
$$

This tells us that the energy E of the three dimensional system does indeed coincide with the energy of the e↵ective one dimensional system that we've reduced to. The e↵ective potential energy is the real potential energy, together with a contribution from the angular kinetic energy.

We already saw in Section 2.1.1 how we can understand qualitative aspects of one dimensional motion simply by plotting the potential energy. Let's play the same game here. We start with the most useful example of a central potential: $V(r) = k/r$, corresponding to an attractive inverse square law for $k > 0$. The e↵ective potential is

$$
V_{\text{eff}} = \frac{k}{r} + \frac{ml^2}{2r^2}
$$

and is drawn in the figure.

The minimum of the e↵ective potential occurs at $r_? = ml^2/k$ and takes the value $V_{e↵}(r_?) = k^2/2ml^2$. The possible forms of the motion can be characterised by their energy E.

- $E = E_{\min} = k^2/2ml^2$: Here the particle sits at the bottom of the well $r_?$ and stays there for all time. However, remember that the particle also has angular velocity, given by $✓ = l/r_?^2$. So although the particle has fixed radial position, it is moving in the angular direction. In other words, the trajectory of the particle is a circular orbit about the origin.

  Notice that the radial position of the minimum depends on the angular momentum l. The higher the angular momentum, the further away the minimum. If there is no angular momentum, and $l = 0$, then $V_{e↵} = V$ and the potential has no minimum. This is telling us the obvious fact that there is no way that r can be constant unless the particle is moving in the ✓ direction. In a similar vein, notice that there is a relationship between the angular velocity ✓ and the size of the orbit, $r_?$, which we get by eliminating l: it is $✓^2 = k/mr_?^3$. We'll come back to this relationship in Section 4.3.2 when we discuss Kepler's laws of planetary motion.

- $E_{\min} < E < 0$: Here the 1d system sits in the dip, oscillating backwards and forwards between two points. Of course, since l = 06 , the particle also has angular velocity in the plane. This describes an orbit in which the radial distance r depends on time. Although it is not yet obvious, we will soon show that for $V = k/r$, this orbit is an ellipse.

  The smallest value of r that the particle reaches is called the *periapsis*. The furthest distance is called the *apoapsis*. Together, these two points are referred to as the *apsides*. In the case of motion around the Sun, the periapsis is called the *perihelion* and the apoapsis the *aphelion*.

- $E > 0$. Now the particle can sit above the horizontal axis. It comes in from infinity, reaches some minimum distance r, then rolls back out to infinity. We will see later that, for the $V = k/r$ potential, this trajectory is hyperbola.

4.2.2 The Stability of Circular Orbits

Consider a general potential $V(r)$. We can ask: when do circular orbits exist? And when are they stable?

The first question is quite easy. Circular orbits exist whenever there exists a solution with l = 06 and $\dot{r} = 0$ for all time. The latter condition means that $\ddot{r} = 0$ which, in turn, requires

$$V'_{\text{eff}}(r_\star) = 0$$

In other words, circular orbits correspond to critical points, $r_?$, of $V_{e↵}$. The orbit is stable if small perturbations return us back to the critical point. This is the same kind of analysis that we did in Section 2.1.2: stability requires that we sit at the minimum of the e↵ective potential. This usually translates to the requirement that

$$V''_{\text{eff}}(r_\star) > 0$$

If this condition holds, small radial deviations from the circular orbit will oscillate about $r_?$ with simple harmonic motion.

Although the criterion for circular orbits is most elegantly expressed in terms of the e↵ective potential, sometimes it's necessary to go back to our original potential $V(r)$. In this language, circular orbits exist at points $r_?$ obeying

$$V'(r_\star) = \frac{ml^2}{r_\star^3}$$

These orbits are stable if

$$V''(r_\star) + \frac{3ml^2}{r_\star^4} = V''(r_\star) + \frac{3}{r_\star}V'(r_\star) > 0 \quad (4.9)$$

We can even go right back to basics and express this in terms of the force (remember that?!), $F(r)$ = V 0(r). A circular orbit is stable if

$$F'(r_\star) + \frac{3}{r^\star}F(r_\star) < 0$$

An Example

Consider a central potential which takes the form

$$V(r) = \quad \frac{k}{r^n} \qquad n \geq 1$$

For what powers of n are the circular orbits stable? By our criterion (4.9), stability requires

$$V'' + \frac{3}{r}V' = \quad \Big(n(n+1) \quad 3n\Big)\frac{k}{r^{n+2}} > 0$$

which holds only for n < 2. We can easily see this pictorially in the figures where we've plotted the e↵ective potential for n = 1 and n = 3.

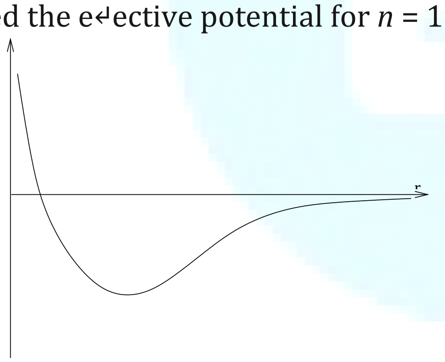


Figure 13: $V_{e↵}$ for V = 1/r

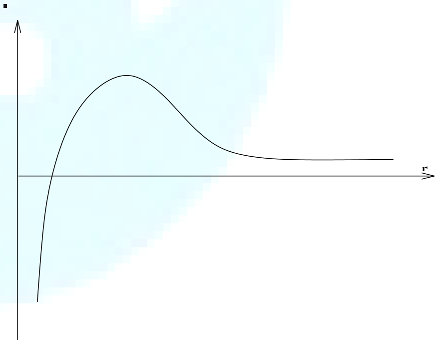


Figure 14: $V_{e↵}$ for V = 1/r^3

Curiously, in a Universe with d spatial dimensions, the law of gravity would be F ←··· $1/r^{d\ 1}$ corresponding to a potential energy V ←··· $1/r^{d\ 2}$. We see that circular planetary orbits are only stable in d < 4 spatial dimensions. Fortunately, this includes our Universe. We should all be feeling very lucky right now.