Derivation of Lagrange's equations from Hamilton's principle

Define a Lagrangian of kinetic and potential energies

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \equiv T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - V(\boldsymbol{q}, t) \tag{29}$$

and define an action potential functional

$S[q(t)]$ — L(q, 4, t) dt (30)

with end points ql — q(tl) and q2 — q(t2). Consider the true path of q(t) from ti to t2 and a variation of the path, öq(t) such that öq(tl) O and öq(t2) O .Hamilton's principle:4

The solution q(t) is an extremum of the action potential $S[q(t)]$

J^{t2} öL(q, t) dt — 0

Substituting the Lagrangian into Hamilton's principle,

$$\int_{t_1}^{t_2} \left\{ \sum_i \left[\frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial T}{\partial q_i} \delta q_i - \frac{\partial V}{\partial q_i} \delta q_i \right] \right\}$$ dt O

We wish to factor out the independent variations (5m, however the first term contains the variation of the derivative, (5äi. If the conditions for admissible variations in position öq fully specify the conditions for admissible variations in velocity (54, the variation and the differentiation can be transposed, d

$\overline{J1} \delta q_i =$ (5äi' (31) dt and we can integrate

the first term by parts,

—öäi — $\left[\frac{\partial T}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i \, dt$

Since öq(tl) O and öq(t2) 0,

$\int_{t_1}^{t_2} \left\{ \sum_i \left[-\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i \right. \right.$ öqi -F —öqi — —öqt dt — 0

Dqz Dqz

The variations öqi must be arbitrary, so the term within the square brackets must be zero for all i.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0.$$