ENTRI

DIFFERENTIABILITY

DIFFERENTIABILITY OF A FUNCTION

Let $f: A \to B$ be a function, then f is said to be differentiable at the point $x = x_0 \in D(f)$ if the limit $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists. If so we denote the derivative of f at a by $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

LEFT AND RIGHT DERIVATIVES

- $f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$.
- $f'(a^+) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

SUFFICIENT CONDITION FOR DIFFERENTIABILITY

Let f be continuous and monotonic at $x = a$, then f is differentiable at a.

RESULT

- Let f, g be differentiable functions, then the functions $\max\{f, g\} = \frac{1}{2}\big((f+g) + |f-g|\big)$ and $\min\{f, g\} = \frac{1}{2}\big((f+g) - |f-g|\big)$ are differentiable except at those points x such $f(x) = g(x)$.
- Let $f(x)$ be a differentiable function, then
  - ❖ $f'(x) > 0 \Rightarrow f$ is strictly Increasing.
  - ❖ $f'(x) \geq 0 \Rightarrow f$ is Increasing.
  - ❖ $f'(x) < 0 \Rightarrow f$ is strictly Decreasing.
  - ❖ $f'(x) \leq 0 \Rightarrow f$ is Decreasing.

LOCAL EXTREMA

Let $f: A \to B$ be a real valued function, then f is said to have a local maxima at $x = a$ if $f(a) \geq f(x)$ for all x in some neighbourhood. $(a-\delta, a+\delta)$, similarly f is said to have a local minima at $x = a$ if $f(a) \leq f(x)$ for all x in some neighbourhood. $(a-\delta, a+\delta)$.

RESULT

- Let f be a twice differentiable function. Then
  - ❖ Suppose $f'(x_0) = 0 \,\& f''(x_0) > 0$, then, x_0 is a local minima.
  - ❖ Suppose $f'(x_0) = 0 \,\& f''(x_0) < 0$, then, x_0 is a local maxima.
  - ❖ Suppose $f'(x_0) = 0 \,\& f''(x_0) = 0$, then, x_0 is a point of inflection.

- Angle between two diffrentiable functions $f \,\&\, g$ at a point (x_0, y_0):

$$\tan\theta = \frac{|m_1 - m_2|}{1 + m_1 m_2}, \qquad m_1 = f'(x_0) \,\&\, m_2 = g'(x_0)$$

ENTRI

CONVEX FUNCTIONS

Let f be a twice diffrentiable on $[a,b]$, then f is said to be convex in $[a,b]$ if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2), \qquad 0 \leq t \leq 1, \qquad x_1x_2 \in [a,b]$$

- f is convex in $[a,b] \Leftrightarrow f''(x) \geq 0$ in $[a,b]$.

ROLLE'S THEOREM

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b), suppose thet $f(a) = f(b)$, then there is an element $c \in (a,b)$ such that $f'(c) = 0$

NOTE

- Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b), suppose that x_0, x_1 are two zeros of f, then $\exists x' \in (x_0, x_1)$ such that $f'(x') = 0$.
- For differentiable function f in an interval, then $\exists$ atlest one zero for f' between any two zeros of f.

LEGRANGE'S MEAN VALUE THEOREM

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

CAUCHY'S GENERALIZED MEAN VALUE THEOREM

Let $f(x)$ be ccontinuous on $[a,b]$ and differentiable on (a,b), suppose $g' \neq 0$ & $g(a) \neq g(b)$ then, $\exists c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

SEQUENCE AND SERIES OF FUNCTIONS

POINTWISE CONVERGENCE

Suppose $\{f_n\}, n = 1,2,3, \ldots$ be a sequence of functions defined on a set E. Then the sequence is said to be converges pointwise to the function $f: E \to \mathbb{R}$ on E if and only if,

$$f(x) = \lim_{n\to\infty} f_n(x), x \in E \; for \; all \; x \in E.$$

UNIFORM CONVERGENCE

We say that a sequence of functions $\{f_n\}, n = 1,2,3, \ldots$ converges uniformly on E to a function f if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon \; for \; all \; x \, \epsilon E.$$

ENTRI

SOME IMPORTANT THEOREMS

CAUCHY CRITERION

The sequence of functions $\{f_n\}$defined on E converges uniformly on E if and only if for every

$\varepsilon > 0$ there exists an integer N such that $n \geq N,\ m \geq N, x \in E$ implies

$|f_m(x) - f_n(x)| \leq \varepsilon.$

UNIFORM CONVERGENCE AND BOUNDEDNESS

If $\{f_n\}$ is a sequence of bounded functions on E and if $f_n \to f$ uniformly on E. then f is bounded on E.

UNIFORM CONVERGENCE AND CONTINUITY

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E. then f is continuous on E.

TEST FOR UNIFORM CONVERGENCE OF SEQUENCES

Let $\{f_n\}$ be a sequence of functions such that $f(x) = \lim_{n\to\infty} f_n(x), x \in [a, b]$. and let $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on [a, b] if and only if $M_n \to 0\ as\ n \to \infty$.

TEST FOR UNIFORM CONVERGENCE OF SERIES

(WEISTRASS M-TEST)

A series of functions $\sum f_n$will converge uniformly (and absolutely) on [a, b] if there exists a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$

$|f_n(x) \leq M, for\ all\ n.$

ABEL'S TEST

If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of x in the interval

[a, b] and $b_n(x)$ is bounded for all values of n and x concerned, and if the series $\sum u_n(x)$is uniformly convergent on [a, b] then so also is the series $\sum b_n(x)u_n(x)$.

RIEMANN INTEGRAL

RIEMANN INTEGRABLE FUNCTION

Let $[a, b]$ be a given interval. A partition P of $[a, b]$ is a finite set of points $x_0, x_1, x_2, \dots, x_n$such that $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$. We write $P = \{x_0, x_1, x_2, \dots, x_n\}$

If P is a partition of $[a, b]$ we write $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$. Let f be a bounded real valued function on $[a, b]$. For P,

$M_i = \sup\{f(x): x_i \leq x \leq x_{i-1}\}$ and $m_i = \inf\{f(x): x_i \leq x \leq x_{i-1}\}$

$U(P,f)=\sum_{i=1}^{n} M_i\Delta x_i$ and $L(P,f)=\sum_{i=1}^{n} m_i\Delta x_i$

$U(P,f)$ and $L(P,f)$ are called the Upper and Lower Reimann Sums for the partition P.

Since f is bounded, there exists real numbers m and M such that $m \le f(x) \le M$ for all $x \in [a,b]$, thus for every partition P of $[a,b]$

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

$$\overline{\int_a^b} f(x)dx = \inf U(P,f)$$

$$\underline{\int_a^b} f(x)dx = \sup L(P,f)$$

These are called the upper and lower Reimann integrals of f over $[a,b]$ respectively.

If the upper and lower Reimann integrals are equal, then f is said to be Reimann integrable or integrable.

NOTES

- Not every bounded function is integrable.
- If a bounded function f is Reimann Integrable on $[a,b]$ then $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = \int_a^b f(x)dx$
- If $f(x) = \begin{cases} 0, when\ x\ is\ rational \\ 1, when\ x\ is\ irrational' \end{cases}$ then f is not integrable on any bounded interval but f is bounded
- Every constant function is integrable

REFINEMENT OF A PARTITION

A partition P_2 of $[a,b]$ is said to be finer than a partition P_1 if $P_2 \supset P_1$. In this case P_2 is a refinement of P_1. Given two partitions P_1 and P_2, the partition $P_1 \cup P_2 = P$ is called their common refinement.

- If P_1 and P_2 are any two partitions of $[a,b]$, then $L(P,f) \le U(P,f)$. That is, an upper sum can never be less than any lower sum.
- Let f be a bounded function on $[a,b]$ and P a partition of $[a,b]$. If P^* is any refinement of P, then
  - $U(P^*,f) \le U(P,f)$
  - $L(P^*,f) \ge L(P,f)$
- If f is continuous and integrable on $[a,b]$, then there exists a number c lying between a and b such that $\int_a^b f(x)dx = (b-a)f(c)$
- If f is bounded and integrable on $[a,b]$ such that $|f(x)| < k \forall x \in [a,b]$, then $\left|\int_a^b f dx\right| \le k|b-a|, k > 0$

REIMANN CRITERION FOR THE INTEGRABILITY OF A BOUNDED FUNCTION

A necessary and sufficient condition for the integrability of a bounded function f is that to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that for every partition P of $[a, b]$ with norm $\mu(P) < \delta, U(P, f) - L(P, f) < \varepsilon$

NOTE

- Every continuous function on $[a, b]$ is integrable on $[a, b]$, but the converse need not be true.
- Every monotonic function on $[a, b]$ is integrable on $[a, b]$
- A bounded function f having only finite number of points of discontinuities in $[a, b]$ is integrable on $[a, b]$
- If a function f is bounded in $[a, b]$ and the set of its points of discontinuity has a finite number of limit points, then f is integrable in $[a, b]$
- If a bounded function f is integrable at $[a, b]$ implies it is integrable on $[a, c]$ and $[c, b]$, $a < c < b$ and $\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$
- If f_1 *and* f_2 are two bounded and integrable functions on $[a, b]$, then $f_1 \pm f_2$ is bounded and integrable on $[a, b]$ and $\int_a^b (f_1 \pm f_2) dx = \int_a^b f_1 dx \pm \int_a^b f_2 dx$
- f is integrable on $[a, b]$ implies f^2 is integrable on $[a, b]$
- If f_1 *and* f_2 are integrable on $[a, b]$, then $f_1 f_2$ is also integrable on $[a, b]$
- If f_1 *and* f_2 are two bounded and integrable functions on $[a, b]$, and let $\lambda > 0$ such that $|f_2(x)| > \lambda \, \forall x \in [a, b]$, then $\frac{f_1}{f_2}$ is bounded and integrable on $[a, b]$
- f is bounded and integrable on $[a, b]$ implies $|f|$ is also bounded and integrable on $[a, b]$ and $\left|\int_a^b f \, dx\right| \leq \int_a^b |f| dx$. The converse need not be true
- A function f is bounded and integrable on $[a, b]$ and there exists a function F such that $F' = f$ on $[a, b]$ and $\int_a^b f dx = F(b) - F(a)$
- If $\int_a^b f dx$ and $\int_a^b g dx$ both exist and f is monotonic on $[a, b]$then there exists some $c \in [a, b]$ such that $\int_a^b f g \, dx = f(a) \int_a^c g dx + f(b) \int_c^b g dx$
- If f is a non negative continuous function on $[a, b]$ such that $\int_a^b f dx = 0$, then $f(x) = 0, \forall x \in [a, b]$
- If f *and* g are two bounded and integrable functions on $[a, b]$, then $\left[\int_a^b f(x) g(x) dx\right]^2 \leq \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx$ and equality holds iff f and g are constants
- Let f be integrable on $[a, b]$ and for each $x \in [a, b]$, let $F(x) = \int_a^b f(t) dt$, then F is uniformly continuous on $[a, b]$

IMPROPER INTEGRALS

IMPROPER INTEGRAL

If the integrand f becomes infinite in the interval $a \leq x \leq b$. f has points of infinite discontinuity in [a, b], or the limits of integration a or b or both become infinite the symbol $\int_a^b f\, dx$ is called an improper integral.

TYPES OF IMPROPER INTEGRALS

1. If $\int_a^t f(x)\, dx$ exists for every number $t \geq a$, then
   $\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x) dx.$
   Provided that limit exists and is finite.
2. If $\int_t^b f(x)\, dx$ exist for every number $t \leq b$ then,
   $\int_{-\infty}^b f(x)\, dx = \lim_{t\to-\infty} \int_t^b f(x) dx.$
   Provided that limit exists and is finite.

The improper integrals $\int_a^\infty f(x)\, dx$ and $\int_{-\infty}^b f(x)\, dx$ are called convergent if the corresponding limit exists and is finite and divergent if the limit does not exists.

3. If both $\int_a^\infty f(x)\, dx$ and $\int_{-\infty}^a f(x)\, dx$ are convergent, then we define,
   $\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx.$

THEOREM

$\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$and divergent if $p \leq 1$.

COMPARISON TEST FOR INTEGRALS

If f and g are continuous functions with $f(x) \geq g(x) \geq 0 \;\; for \;\; x \geq a$ then,

1. If $\int_a^\infty f(x)\, dx$ is convergent then $\int_a^\infty g(x)\, dx$ is convergent.
2. $\int_a^\infty g(x)\, dx$ is divergent then, $\int_a^\infty f(x)\, dx$ is divergent.