Postulates of quantum mechanics
Part 5
Expectation Value & Eigen Value

Postulate 3 ; Expectation Value

When a system is in a state described by a wave function Ψ, the expectation value of any observable A is given by

$$\langle A \rangle = \int_{-\infty}^{\infty} \Psi^{*} \mathrm{A}\, \Psi \, d\tau$$

Where A in the integral is the operator associated with the observable A. in the above equation Ψ is normalized. If the wave function is not normalized.

$$\langle A \rangle = \frac{\int_{-\infty}^{\infty} \Psi^{*} \mathrm{A}\, \Psi \, d\tau}{\int_{-\infty}^{\infty} \Psi^{*} \Psi \, d\tau}$$

Postulate 4 ; Eigenvalues

The possible values which a measurement of an observable, whose operator is A, can give are the eigenvalues a_i of the equation.

$$\mathrm{A}\Psi_i = a_i \Psi_i \qquad \Psi_i = 1, 2, 3, \ldots\ldots, n$$

When an experiment is performed to determine the value of an observable A in a particular state, the value we expect in the measurement is its eigenvalue. In other word, the eigenvalues of an operator are the only experimentally measurable quantities. Hence the eigenvalues always give a real number. We have already seen that the eigenvalues of Hermitian operators are real. Therefore operators associated with physical quantities must be Hermitian.

If an operator $\hat{A}$ act on Ψ then

$$\hat{A}\Psi = \lambda\Psi$$

Here λ is called eigenvalue and Ψ is known as eigen function.

Example ; $\Psi = e^{2x}$ and $\hat{A} = \frac{d^2}{dx^2}$

$$\hat{A}\Psi = \frac{d^2(e^{2x})}{dx^2}$$

$$= 4\, e^{2x}$$

$$\lambda = 4$$

Eigen function = e^{2x}

Note

We have the expectation value of A

$$\langle A \rangle = \frac{\int_{-\infty}^{\infty} \Psi^* A\, \Psi\, d\tau}{\int_{-\infty}^{\infty} \Psi^* \Psi\, d\tau}$$

Using the property of eigen value we can substitute

$$\hat{A}\Psi = \lambda\Psi$$

$$\langle A \rangle = \frac{\int_{-\infty}^{\infty} \Psi^* \lambda\, \Psi\, d\tau}{\int_{-\infty}^{\infty} \Psi^* \Psi\, d\tau}$$

$$\text{But } \int_{-\infty}^{\infty} \Psi^* \Psi\, d\tau = 1$$

$$\langle A \rangle = \lambda \int_{-\infty}^{\infty} \Psi^* \Psi\, d\tau$$

$$\langle A \rangle = \lambda$$

Energy eigenvalue

$$\hat{E}\,\Psi = E_n \Psi$$

$$E_n = \frac{E_1}{n^2}$$

E_n *is the* **energy of the** n^{th} **state n = 1, 2, 3, ..., n**

E = K + V

$$I\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}$$

Postulate 5 ; Time Development of a Quantum System

The first - four postulates describe the concept of a quantum system at a given instant of time whereas the fifth one deals with the time development of a system. The time development can be studied systematically with the help of equations of motion which could be differential equations of the physical variables describing the system. The state or wave function Ψ (r,t) which describes the state of the system as possible may be brought into the picture.

Postulate ; the time development of a quantum system can be described by the evolution of state function in time by the time - dependent schrödinger equation.

$$i\hbar \frac{\partial \Psi(r,t)}{\partial t} = H\Psi(r,t)$$

Where H is the Hamiltonian operator of the system which is independent of time.

Consider a quantum mechanical system when t = 0 the wave function will be Ψ(0) and when t = t what will be the wave function Ψ(t) = ?

When an energy operator $\hat{E}$ acts on the wave function Ψ

$$\hat{E}\Psi = E_n\Psi$$

$$i\hbar \frac{\partial \Psi}{\partial t} = E_n\Psi$$

The above equation is in varable seprable form

$$\frac{\partial \Psi}{\Psi} = \frac{1}{i\hbar} E_n \, \partial t$$

$$\frac{\partial \Psi}{\Psi} = \frac{-i\,E_n}{\hbar} \, \partial t$$

$$\int_{\Psi(0)}^{\Psi(t)} \frac{\partial \Psi}{\Psi} = \frac{-i\,E_n}{\hbar} \int_0^t \partial t$$

$$[\ln \Psi\,]_{\Psi(0)}^{\Psi(t)} = \frac{-i\,E_n}{\hbar}\, t$$

$$\ln\left[\frac{\Psi(t)}{\Psi(0)}\right] = \frac{-i\,E_n}{\hbar}\, t$$

$$\frac{\Psi(t)}{\Psi(0)} = e^{\frac{-iE_n t}{\hbar}}$$

$$\Psi(t) = \Psi(0)\; e^{\frac{-iE_n t}{\hbar}}$$