ENTRI

Wave Properties of Particles
Part 4
Group velocity and phase velocity

Formula for Phase and Group velocities

- In terms of angular frequency ω and wave vector K.

$$V_p = \frac{\omega}{K} \qquad \text{where } K = \frac{2\pi}{\lambda}$$

$$V_g = \frac{d\omega}{dK}$$

- In terms of energy and momentum.

$$V_p = \frac{\omega\hbar}{K\hbar}$$

ħ is the reduced planck's constant ; $\hbar = \frac{h}{2\pi} = 1.05\times10^{-34}$ js

$$\hbar\omega = \frac{h}{2\pi} \times 2\pi\nu = h\nu$$

$$\hbar k = \frac{h}{2\pi} \times \frac{2\pi}{\lambda} = \frac{h}{\lambda}$$

$$V_p = \frac{E}{P}$$

$$V_g = \frac{dE}{dP}$$

- In terms of wavelength and frequency.

$$V_p = \nu\lambda$$

$$V_g = \frac{d\omega}{dK} = \frac{d(2\pi\nu)}{d\left(\frac{2\pi}{\lambda}\right)}$$

$$V_g = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)} = -\lambda^2 \frac{d\nu}{d\lambda}$$

- Group Velocity = Velocity of the body (V_g) = V

Relation between V_g and V_p

$$V_g = V_p + k\frac{dV_p}{dk}$$

- **In terms of Frequency and Wavelength**

$$V_p = \nu\lambda$$

$$V_g = -\lambda^2 \frac{d\nu}{d\lambda}$$

Problems

1. Find the relation between V_g and V_p for a relativistic particle. Or find the product of V_g and V_p for a relativistic particle.

Solution :

$$E^2 = P^2C^2 + m_0{}^2C^4 \quad \text{----------- (1)}$$

$$V_p = \frac{E}{P}$$

$$V_g = \frac{dE}{dP}$$

Differentiating equ (1) $\Longrightarrow$ $E\frac{dE}{dP} = 2PC^2 + 0$

$$\frac{dE}{dP} = \frac{PC^2}{E} = V_g$$

$$V_pV_g = \frac{E}{P} \times \frac{PC^2}{E} = C^2$$

2. The wave number k and frequency ω of a wave are related by the dispersion relation $\omega^2 = \alpha k + \beta k^3$, α and β are positive constants. The wave number for which phase velocity equals group velocity is

(a) $3\sqrt{\frac{\alpha}{\beta}}$ (b) $\sqrt{\frac{\alpha}{\beta}}$ (c) $\frac{1}{2}\sqrt{\frac{\alpha}{\beta}}$ (d) $\frac{1}{3}\sqrt{\frac{\alpha}{\beta}}$

Solution ; (b) $\sqrt{\frac{\alpha}{\beta}}$

Given $\omega^2 = \alpha k + \beta k^3$

We have $V_p = \frac{\omega}{k}$

$$V_g = \frac{d\omega}{dk}$$

$$\omega = \sqrt{\alpha k + \beta k^3}$$

$$V_p = \frac{\sqrt{\alpha k+\beta k^3}}{k}$$

$$V_g = \frac{d\sqrt{\alpha k+\beta k^3}}{dK}$$

$$= \frac{\alpha+3\beta K^2}{2\sqrt{\alpha k+\beta k^3}}$$

According to question V_p and V_g

$$\frac{\sqrt{\alpha k+\beta k^3}}{k} = \frac{\alpha+3\beta K^2}{2\sqrt{\alpha k+\beta k^3}}$$

$$2(\alpha k+\beta k^3) = \alpha K + 3\alpha K^3$$

$$2\alpha K + 2\beta K^3 = \alpha K + 3\alpha K^3$$

$$\alpha K = \beta K^3$$

$$\alpha = \beta K^2$$

$$K^2 = \frac{\alpha}{\beta}$$

$$K = \sqrt{\frac{\alpha}{\beta}}$$

3. A particle of rest mass m_0 is moving uniformly in a straight line with a relativistic velocity βc, where c is the velocity of light in vacuum and $0 < \beta < 1$. The phase velocity of the de Broglie wave associated with the particle is ________

a) βc

b) $\frac{c}{\beta}$

c) c

d) $\frac{c}{\beta^2}$

Solution : $\frac{c}{\beta}$

Given , V= βc ; also group velocity is the velocity of the particle.

$$V_g = V = \beta c$$

For a relativistic particle $V_g V_p = c^2$

$$V_p = \frac{c^2}{v_g} = \frac{c^2}{\beta c} = \frac{c}{\beta}$$

4. **The angular frequency ω of deep - water waves varies as the inverse square root of the wavelength. Which of the following is the relation between group velocity and phase velocity ?**

a) $V_g = \frac{V_p}{2}$

b) $V_g = V_p$

C) $V_g = 2V_P$

d) $V_g = V_P\lambda$

Solution :

$$\text{Given } \omega \propto \frac{1}{\sqrt{\lambda}} \quad ; \quad \omega = \frac{a}{\sqrt{\lambda}}$$

$$K = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{K} \qquad \omega = \frac{a}{\sqrt{2\pi}}$$

$$V_p = \frac{\omega}{K} = \frac{a}{\sqrt{2\pi}} \frac{\sqrt{K}}{K} = \frac{a}{\sqrt{2\pi K}}$$

$$V_g = \frac{d\omega}{dK} = \frac{d}{dK}\left[\frac{a}{\sqrt{2\pi K}} \sqrt{k}\right]$$

$$V_g = \frac{a}{\sqrt{2\pi}} \frac{1}{2\sqrt{K}}$$

$$= \frac{a}{2} \frac{1}{\sqrt{2\pi K}}$$

$$V_g = \frac{V_p}{2}$$

5. **The wave velocity, which is associated with a particle is given by $V_p = \frac{\alpha}{\lambda^2}$ in any medium. Then the particle velocity is related by wave velocity as**

a) $V_g = V_p$

b) $V_g = 2V_p$

c) $V_g = 3V_p$

d) $V_g = \frac{V_p}{2}$

Solution ; (c) $V_g = 3V_p$

Given $V_p = \frac{\alpha}{\lambda^2}$

$V_g = V_p - \lambda \frac{dV_P}{d\lambda}$

$= \frac{\alpha}{\lambda^2} - \lambda \frac{d(\propto \lambda^{-2})}{d\lambda}$

$= \frac{\alpha}{\lambda^2} - \lambda \propto \times -2\lambda^{-3}$

$= \frac{\alpha}{\lambda^2} + \frac{2\propto\lambda}{\lambda^3} = \frac{\alpha}{\lambda^2} + \frac{2\propto}{\lambda^2}$

$V_g = \frac{3\propto}{\lambda^2} = 3V_P$

6. The waves propagating in a medium obeys the dispersion relation $\omega = (V^4K^4 + {\omega_0}^4)^{\frac{1}{4}}$. Find the ratio of phase velocity to group velocity .

a) $(V^4K^4 + {\omega_0}^4)^4$ b) $\left(1 + \frac{\omega_0}{KV}\right)^4$

c) $1 + \left(\frac{\omega_0}{KV}\right)^4$ d) $\left(1 - \frac{\omega_0}{KV}\right)^4$

Solution : c) $1 + \left(\frac{\omega_0}{KV}\right)^4$

Given $\omega = (V^4K^4 + {\omega_0}^4)^{\frac{1}{4}}$

$\frac{V_P}{V_g} = ?$

$V_P = \frac{(V^4K^4 + {\omega_0}^4)^{\frac{1}{4}}}{K}$

$V_g = \frac{1}{4} (V^4K^4 + {\omega_0}^4)^{\frac{-3}{4}} \times 4V^4K^3$

$= \frac{V^4K^3}{(V^4K^4 + {\omega_0}^4)^{\frac{3}{4}}}$

$\frac{V_P}{V_g} = \frac{(V^4K^4 + {\omega_0}^4)^{\frac{1}{4}}}{K} \times \frac{(V^4K^4 + {\omega_0}^4)^{\frac{3}{4}}}{V^4K^3}$

$$= \frac{(V^4K^4+\omega_0{}^4)^1}{V^4K^4}$$

$$\frac{V_P}{V_g} = 1 + \left(\frac{\omega_0}{KV}\right)^4$$