HAMILTON'S PRINCIPLE

In our previous derivation of Lagrange's equations we started from the Newtonian vector equations of motion and, via D'Alembert's Principle, changed coordinates to generalised coordinates ending up with Lagrange's equations of motion. There is another way to express the basic laws of mechanics in a single statement which is equivalent to Lagrange's equations; this statement is called Hamilton's Principle or sometimes the Principle of Least Action. It differs fundamentally from the differential equation formulation in that it refers to the entire history of a system's motion between two distinct times.

To introduce the idea we first define the *action* for a system of n degrees of freedom by

$$S[q_1, q_2, \ldots, q_n] = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)\, dt \, ,$$

where t_1 and t_2 are respectively an initial and a final time, and L is the Lagrangian for the system which depends on the n generalised coordinates $q_i(t)$, the n generalised velocities $\dot{q}_i(t)$ and possibly the time (if there are explicitly time- dependent external fields). Because we integrate over a finite time interval we see that S depends on the entire trajectory of the system in configuration space between t_1 and t_2. Because S depends on the entire trajectory rather than just on the values of the coordinates and velocities at one instant we say that S is a *functional* of the trajectories $q_i(t)$.

Hamilton's Principle can be stated thus: Amongst all the possible trajectories $q_i(t)$ which take the system from initial configuration $q_{i1} = q_i(t_1)$ at time t_1 to final configuration $q_{i2} = q_i(t_2)$ at time t_2, the physical trajectory is the one which makes the action an extremum (usually a minimum but could be a saddle point).

Hamilton's Principle is stated in terms of the action S, which is a scalar quantity that shares all the invariances of the Lagrangian L, and which is independent of any particular choice of generalised coordinates. However, we can show that Hamilton's Principle implies that the trajectory which minimizes the action is the one that also satisfies Lagrange's equations so we see that the two formulations are equivalent. For simplicity we consider a system with one degree of freedom so that there is only one generalised coordinate $q(t)$. The system starts at position $q_1 = q(t_1)$ and finishes at position $q_2 = q(t_2)$. We can denote the physical trajectory which joins these initial and final positions by $\bar{q}(t)$ which satisfies $\bar{q}(t_1) = q_1$ and $\bar{q}(t_2) = q_2$. To implement Hamilton's Principle we now have to consider "all possible" sufficiently smooth trajectories $q(t)$ different from $\bar{q}(t)$ that start and end at q_1, q_2 respectively, and then show that $S[q(t)]$ has an extremum at $\bar{q}(t)$. We can express "all possible" other trajectories $q(t)$ in terms of their deviation $\eta(t)$ from the physical trajectory $\bar{q}(t)$ as

$$q(t) = \bar{q}(t) + \eta(t) \quad ,$$

where at the endpoints we must have $\epsilon(t_1) = \epsilon(t_2) = 0$ since all paths must start at q_1 and finish at q_2. To find a minimum, or an extremum, it is sufficient to consider "nearby" paths such that $\epsilon(t)$ is small. Thus we can expand the action in powers of ϵ. To do this we first Taylor expand the Lagrangian as

$$L(q, \dot{q}, t) = L(\bar{q} + \epsilon, \dot{\bar{q}} + \dot{\epsilon}, t) = L(\bar{q}, \dot{\bar{q}}, t) + \epsilon \frac{\partial L}{\partial \bar{q}} + \dot{\epsilon} \frac{\partial L}{\partial \dot{\bar{q}}} + \ldots ,$$

which gives for the action

$$S[q] = S[\bar{q} + \epsilon] = \int_{t_1}^{t_2} L(\bar{q} + \epsilon, \dot{\bar{q}} + \dot{\epsilon}, t)\, dt = \int_{t_1}^{t_2} L(\bar{q}, \dot{\bar{q}}, t)\, dt + \int_{t_1}^{t_2} \epsilon \frac{\partial L}{\partial \bar{q}}\, dt + \int_{t_1}^{t_2} \dot{\epsilon} \frac{\partial L}{\partial \dot{\bar{q}}}\, dt + \ldots .$$

We can rearrange this as

$$S[\bar{q} + \epsilon] - S[\bar{q}] = \int_{t_1}^{t_2} \epsilon \frac{\partial L}{\partial \bar{q}}\, dt + \int_{t_1}^{t_2} \dot{\epsilon} \frac{\partial L}{\partial \dot{\bar{q}}}\, dt + \ldots ,$$

and then in the second integral perform an interation by parts

$$\int_{t_1}^{t_2} \dot{\epsilon} \frac{\partial L}{\partial \dot{\bar{q}}}\, dt = \left[\epsilon(t) \frac{\partial L}{\partial \dot{\bar{q}}} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \epsilon \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\bar{q}}} \right) dt .$$

Since ϵ vanishes at the two endpoints t_1 and t_2, there is no contribution from these endpoints and we have

$$\delta S = S[\bar{q} + \epsilon] - S[\bar{q}] = \int_{t_1}^{t_2} \epsilon \left(\frac{\partial L}{\partial \bar{q}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\bar{q}}} \right) \right) dt + \ldots .$$

This expression for δS, which is first order in the small quantity ϵ, is called the first variation of S. It plays the role for functionals that a first derivative plays for an ordinary function. In particular, we see that if S is to have a minimum or an extremum at $q(t)$, this first variation must vanish identically *for any choice of* $\epsilon(t)$. That in turn will be possible only if, for all times t between t_1 and t_2, $t_1 \le t \le t_2$, the coefficient of $\epsilon(t)$ in the integral above vanishes, namely that

$$\frac{\partial L}{\partial \bar{q}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\bar{q}}} \right) = 0 .$$

However, this is just Lagrange's equation for $q(t)$, showing that Hamilton's principle predicts the same mechanical trajectory that Lagrange's equations would predict. For more than one degree of freedom the same argument goes through with appropriate subscripts added such as $\epsilon_i(t)$, etc.

HAMILTONIAN MECHANICS

In addition to showing us how Lagrange's equations can be derived from the action principle, Hamilton showed us an entirely novel way to reformulate the basic laws of mechanics so that ideas of geometry and topology can come to the fore in

analyzing mechanical motion. As a by-product he gave us a form of classical mechanics which can be transformed directly into quantum mechanics.

To understand the changed point of view which takes us to Hamiltonian mechanics, we first recall that in Lagrangian mechanics the positions q_i and velocities $\dot{q}_i$ are the fundamental variables and the Lagrangian itself is a function of these variables and of the time if there are explicitly time dependent fields present, $L = L(q_i, \dot{q}_i, t)$. If we make a small change in all these variables then L changes as

$$dL = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \, ,$$

or, using the definition of generalised momentum, $p_i = \partial L / \partial \dot{q}_i$, and the Lagrange equations of motion, $\partial L / \partial q_i = \dot{p}_i$,

$$dL = \sum_{i=1}^{n} \dot{p}_i dq_i + \sum_{i=1}^{n} p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \, .$$

Recall also that in Lagrangian mechanics the energy E is defined as

$$E = \sum_{i=1}^{n} p_i \dot{q}_i - L \, ,$$

and so we would expect it to depend on positions and velocities as well. However, if we calculate the change in E, analogous to the change in L above, we get something unexpected,

$$dE = \sum_{i=1}^{n} p_i d\dot{q}_i + \sum_{i=1}^{n} \dot{q}_i dp_i - dL = \sum_{i=1}^{n} p_i d\dot{q}_i + \sum_{i=1}^{n} \dot{q}_i dp_i - \sum_{i=1}^{n} \dot{p}_i dq_i - \sum_{i=1}^{n} p_i d\dot{q}_i - \frac{\partial L}{\partial t} dt \, ,$$

$$dE = -\sum_{i=1}^{n} \dot{p}_i dq_i + \sum_{i=1}^{n} \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt \, ,$$

where two terms have cancelled out. Unlike the expression for dL above which says that L is in fact a function of the independent variables q_i and $\dot{q}_i$, the expression for dE says that mathematically we should regard E as a function of the independent variables q_i, p_i and t.

How can we realize this explicitly? In the Lagrangian point of view, $p_i = \partial L / \partial \dot{q}_i = p_i(q_i, \dot{q}_i, t)$, so that p_i is a secondary quantity which is a function of q_i, $\dot{q}_i t$. However, if we can invert this equation and find $\dot{q}_i$ as a function of q_i, p_i and t, then we can eliminate all velocities from the energy E in favour of positions q_i and momenta p_i. If we have carried out this transformation, we define the resulting energy function as the Hamiltonian H. In other words $E(q_i, \dot{q}_i, t) \to H(q_i, p_i, t)$. We then write the infinitesimal change in the energy as

$$dH(q_i, p_i, t) = -\sum_{i=1}^{n} \dot{p}_i dq_i + \sum_{i=1}^{n} \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt \, .$$

From this we can read off (analogous to the way that we use the thermodynamic identity in thermal physics) that

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad .$$

If there is no explicit dependence on time in L then there will be no explicit dependence on time in H and we have the $2n$ first order equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , i = 1, 2, \ldots, n \quad .$$

as the fundamental equations of motion for the system. These are called the Hamilton equations of motion, or sometimes, the canonical equations of motion. In this point of view, the energy function H generates the equations of motion and the variables q_i, p_i appear on an equal footing rather than the momenta being seen as secondary quantities as in the Lagrangian picture.

The origin of conservation laws in the Hamiltonian formulation lies in the symmetries of the system, just as in the Lagrangian framework. To see this, note that if the Hamiltonian does not depend on a particular q_i (i.e., q_i is a cyclic variable), then from the Hamilton equation $\dot{p}_i = -\partial H/\partial q_i = 0$ we have that the corresponding momentum is conserved. For the energy itself we calculate

$$\frac{dH(q_i, p_i, t)}{dt} = \sum_{i=1}^{n} \frac{dq_i}{dt} \frac{\partial H}{\partial q_i} + \sum_{i=1}^{n} \frac{dp_i}{dt} \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t} \quad ,$$

and, using the Hamilton equations of motion to express $\dot{q}_i$ and $\dot{p}_i$,

$$\frac{dH(q_i, p_i, t)}{dt} = \sum_{i=1}^{n} [-\dot{q}_i \dot{p}_i + \dot{p}_i \dot{q}_i] + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \quad .$$

Thus if there is no explicit time dependence ($\partial H/\partial t = 0$) in the Hamiltonian (i.e., we have time translation symmetry) the energy (which is H) is conserved as well.

PHASE SPACE

In the Lagrangian picture we talked about *configuration space*, the n-dimensional space whose coordinates are the generalised coordinates q_i for $i = 1,2,...,n$. In the Hamiltonian picture, since both the q_i and p_i appear on an equal footing, we introduce a $2n$-dimensional space whose coordinates are the q_i and p_i. This space we call *phase space*. A single point in phase space specifies completely the position and momentum of every particle in the system, that is, a single point completely specifies the microscopic state of the system. If we specify such a point at time $t = 0$, we can use the associated $q_i(0)$ and $p_i(0)$ as initial conditions for Hamilton's equations and if we solve these equations we find $q_i(t)$ and $p_i(t)$ for all later (and earlier!) times. As a function of time the phase point traces out a *phase trajectory* which describes the entire history of the system. Since each point in phase space is a possible initial state, we can say

that the phase trajectories fill phase space, but no two trajectories can ever intersect because the solution to Hamilton's equations is uniquely specified by the initial conditions. As a result, we can consider that under the Hamiltonian time evolution, the entire phase space flows like a liquid. Using Hamilton's equations of motion we can prove that the volume of any region of phase space is constant under this evolution (Liouville's theorem) although the shape of the region may distort in an unbelievably complicated way.

As an example, consider the simple harmonic oscillator with one degree of freedom. The Lagrangian L and the energy E are respectively

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \quad , \quad E = p\dot{q} - L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 \quad ,$$

where $p = \partial L/\partial\dot{q} = m\dot{q}$. To form the Hamiltonian we express $\dot{q}$ in terms of p, $\dot{q} = p/m$ and substitute in E to get

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

with two Hamilton equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} = -kq \quad .$$

To find the phase trajectories we can either solve Hamilton's equations for $q(t)$ and $p(t)$ and plot the curve parametrically using time as the parameter or we can remember that, with no explicit time dependence, the Hamiltonian is constant on any phase trajectory and we can get the equation of the trajectory by solving $H(q,p) = constant$ for p as a function of q. Either way we find families of ellipses as discussed in the lecture.

As a second example of how to set up a Hamiltonian description consider the spherical pendulum discussed in an earlier summary sheet which has two degrees of freedom. The two convenient generalised coordinates are the spherical polar angles θ and φ. The Lagrangian was

$$L = T - V = \frac{m}{2}\left(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2\right) - mga\cos\theta \quad ,$$

and the energy E was

$$E = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = T + V = \frac{m}{2}\left(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2\right) + mga\cos\theta \quad .$$

The generalised momenta are

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta} \quad , \quad p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ma^2\sin^2\theta\dot{\phi} \quad .$$

If we invert these to express the velocities in terms of the momenta,

$$\dot{\theta} = \frac{p_\theta}{ma^2} \quad , \quad \dot{\phi} = \frac{p_\phi}{ma^2\sin^2\theta} \quad ,$$

and then eliminate the velocities from the energy E, we get the Hamiltonian

$$H(\theta, \phi, p_\theta, p_\phi) = \frac{p_\theta{}^2}{2ma^2} + \frac{p_\phi{}^2}{2ma^2 \sin^2 \theta} + mga \cos \theta \ .$$

We observe that φ is a cyclic variable so from the Hamilton equations we have that p_φ is conserved. The Hamiltonian then effectively reduces to a function of θ and p_θ only and we could draw phase trajectories in the reduced two dimensional phase space with coordinates θ, p_θ.