LEBESGUE MEASURE

MEASURABLE SETS

- Let $[a, b]$ be a closed bounded interval in R. If A is any non-empty bounded open subset of $[a, b]$ then, the length of A is defined as the sum of the lengths of all disjoint open intervals $I_k, k = 1,2,3, \dots$ such that $A \subseteq I_k$ and is denoted by $l(A)$.
- If $A_1$and A_2 are two bounded open sets and $A_1 \subseteq A_2$ then $l(A_1) \leq l(A_2)$.
- If B is any closed subset of the interval [a, b], the complement $C(B)$ of B relative to any open subset F of $[a, b]$ containing B is $F - B = F \cap C(B)$ and is open in $[a, b]$. Then, the length of B is defined as $l(B) = l(F) - l(F - B)$

OUTER MEASURE

Let $A \subseteq [a, b]$ be any bounded subset of R. The outer measure of A denoted by m^*A is defined as $m^*A = l(F)$, where the infimum is taken over all open sets F which contain A.

F being open, can be expressed as a countable union of open intervals $I_n, n = 1,2,3, \dots$ such that $A \subseteq F \subseteq I_n$. For each such countable collection containing A, consider the sum of the lengths of the intervals in that collection. Then, the outer measure of A is defined as $m^*A = \sum_n \quad l(I_n)$.

NOTE

- Measure of any bounded set is non-negative and finite.
- For any empty set ϕ, $m^*\phi = 0$.
- If $A \subseteq B$ then, $m^*A \leq m^*B$.

INNER MEASURE

Let $A \subseteq [a, b]$ be any bounded subset of R. The inner measure of A denoted by m_*A is defined as $m_*A = sup\ sup\ l(F)$, where the supremum is taken over all open sets F which contain A.

F being open, can be expressed as a countable union of open intervals $I_n, n = 1,2,3, \dots$ such that $A \subseteq F \subseteq I_n$. For each such countable collection containing A, consider the sum of the lengths of the intervals in that collection. Then, the inner measure of A is defined as $m_*A = \sum_n \quad l(I_n)$.

DEFINITION

A set $A \subseteq [a, b]$ is said to be measurable if $m^*A = m_*A$. Then, we define mA, the measure of A as,

$$mA = m^*A = m_*A$$

SOME THEOREMS

- For every set A $m^*A = m_*A$.
- If $A \subseteq [a, b]$, then $m^*A + m_*C(A) = b - a$.
- A subset $A \subseteq [a, b]$ is measurable if and only if $m^*A + m_*C(A) \leq b - a$.

- If $A_1$and A_2 are measurable subsets of [a, b] then both $A_1 \cup A_2$ and $A_1 \cap A_2$ are measurable and $mA_1 + mA_2 = m(A_1 \cup A_2) + m(A_1 \cap A_2)$.
- If $A_1, A_2, A_3, \dots, A_n$are pairwise disjoint measurable subsets of [a, b] then $\cup_{n=1}^{\infty} A_n$ is measurable and $m(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \quad mA_n$.
- If $E_1$and E_2 are measurable then $E_1 \cup E_2$ is measurable. E_1 being measurable, by definition, for each set A, $m^*A = m^*\ (A \cap E_1) + m^*(A \cap C(E_1))$.

SETS OF MEASURE ZERO

A subset of R is said to be a set of measure zero if for any $\varepsilon > 0$ there exist a sequence of bounded open intervals $I_1, I_2, I_3, \dots$ such that

1. $A \subseteq I_n$
2. $\sum_{n=1}^{\infty} \quad l(I_n) \le \varepsilon$

THEOREM

The following statements regarding the set E are equivalent:

1. E is measurable
2. For all $\varepsilon > 0$ there exist an open set O $\supseteq E$ such that $m^*(O - E) \le \varepsilon$.
3. There exist G a G_δ set $G \supseteq E$ such that $m^*(G - E) = 0$. (A set G is said to be G_δ if $G = \cap_{i=1}^{\infty} G_i$, each G_i is an open set)
4. For all $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $m^*(E - F) \le \varepsilon$.
5. There exists $F\ an\ F_\sigma\ set\ F \subseteq E$ such that $m^*(E - F) = 0$. (A set F is said to be $F_\sigma$if $F = \cup_{i=1}^{\infty} F_i\ ,\ each\ F_i\ is\ a\ closed\ set$)

MEASURABLE FUNCTIONS

Let f be a function defined on $[a, b]$. we call f to be a measurable function if for each $\alpha \in R$, the set $\{x: f(x) > \alpha\}$ is a measurable set.

EXAMPLE

- Constant functions are measurable.

NOTES

- The function f on [a, b] is measurable if and only if any one of the following conditions hold:
    1. $\{x: f(x) > \alpha\}$ is measurable set for every real α.
    2. $\{x: f(x) \ge \alpha\}$ is measurable set for every real α.
    3. $\{x: f(x) < \alpha\}$ is measurable set for every real α.
    4. $\{x: f(x) \le \alpha\}$ is measurable set for every real α.
- If f is measurable, then $|f|$ is measurable.
- If f is a measurable function on $[a, b]$ and $k \in R$, then $f + k$ and kf are measurable.
- If $f_1\ and\ f_2$ are measurable on [a, b] then so are, $f_1 +\ f_2$, $f_1 -\ f_2$, $f_1 f_2$ and $\frac{f_1}{f_2}$ provided $f_2 \ne 0$ on $[a, b]$.

- Every continuous function is measurable.

LEBESGUE INTEGRAL

LEBESGUE INTEGRAL

MEASURABLE PARTITION

Let f be any bounded function on $[a, b]$ and let $P = (A_1, A_2, \dots, A_n)$ be any partition of [a, b], where $A_1, A_2, \dots, A_n$ are measurable subsets of [a, b] so that

$$\cup_{i=1}^{n} A_i = [a, b] \; and \; m(A_i \cap A_j) = 0 \; for \; i \neq j$$

Such a partition of [a, b] is called a measurable partition of [a, b].

UPPER LEBESGUE INTEGRAL

We define,

$$U(P, f) = \sum_{i=1}^{n} \quad (f(x)) \;\; mA_i$$

$$L(P, f) = \sum_{i=1}^{n} \quad (f(x)) \;\; mA_i$$

As the upper and lower Lebesgue sums of the function f corresponding to the partition $P(A_1, A_2, \dots, A_n)$ of $[a, b]$.

Obviously $U(P, f) \geq \; L(P, f)$ for every partition P.

The infimum of the set of all upper Lebesgue sums is called the upper Lebesgue integral denoted as:

$$L \int_{a}^{\underline{b}} \quad f \, dx = inf \; U(P, f) \; \forall \; partotions \; P$$

LOWER LEBESGUE INTEGRAL

Similarly,

The supremum of the set of all lower Lebesgue integral is called the lower Lebesgue integral denoted as:

$$L \int_{\underline{a}}^{b} \quad f \, dx = sup \; L(P, f) \; \forall \; partotions \; P$$

DEFINITION

Let f be a bounded function on $[a, b]$. then for any two measurable partitions P_1, P_2 of $[a, b]$, we have,

$$L\int_{\underline{a}}^{b} f\,dx = L\int_{a}^{\underline{b}} f\,dx$$

In this case we define,

$$L\int_{a}^{b} f\,dx = L\int_{\underline{a}}^{b} f\,dx = L\int_{a}^{\underline{b}} f\,dx$$

The fact that f is Lebesgue integrable, we express by writing $f \in L[a,b]$.

THEOREMS

- Every bounded Riemann integrable function over [a, b] is Lebesgue integrable and the two integrals are equal.
- A necessary and sufficient condition for a bounded function to be Lebesgue Integrable over [a, b] is that for each given $\varepsilon > 0$ there exists a measurable partition $P\ of\ [a,b]$ such that
$$U(P,f) - L(P,f) < \varepsilon$$
- Every bounded measurable function on [a, b] is Lebesgue integrable.

PROPERTIES

The following properties hold for a bounded Lebesgue integrable function f on [a, b].

1. If $a < c < b$ then f is Lebesgue integrable on [a, c] as well as [c, b] and
$$\int_a^b f = \int_a^c f + \int_c^b f$$
2. If k is any scalar, $k \in R$, then kf is Lebesgue integrable and
$$\int_a^b kf = k\int_a^b f$$
3. If $f = f_1 + f_2$, where f_1, f_2 are Lebesgue integrable on [a, b], then f is Lebesgue integrable on [a, b] and
$$\int_a^b f = \int_a^b f_1 + \int_a^b f_2$$
4. If A_k is a (finite or infinite) sequence of disjoint measurable subsets of [a, b] whose union A has finite measure, then
$$\int_A f = \sum_k \int_{A_k} f$$

THEOREMS

- If f and g are bounded functions and Lebesgue integrable over $[a,b]$ and if
  1. $f(x) \geq 0 \ \ a.e \ \ on\ [a,b] \ \ then\ \int_a^b f(x)dx \geq 0$
  2. $f(x) \leq g(x)\ a.e\ \ on\ [a,b], then\ \int_a^b f\,dx \leq \int_a^b g\,dx$
- If a bounded function f is Lebesgue integrable on [a, b] then $|f|$ is Lebesgue integrable over [a, b]. Moreover, if f is Lebesgue integrable, then

$$|\int_a^b f| \leq \int_a^b |f|$$

- A necessary and sufficient condition for a bounded function f defined on a measurable set A with finite measure to be measurable, is that $\int_A \psi(x)\,dx = \int_A \phi(x)\,dx$ for all simple functions ϕ and ψ on A.
- If f and g are non-negative bounded measurable functions defined on a set $A \subseteq [a,b]$ of finite measure, then
    1. $\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$ $where\ \alpha, \beta\ are\ any\ real\ numbers.$
    2. $if\ f = g\ \ a.e\ then\ \int_A f = \int_A g$
    3. If $f \geq 0\ \ a.e\ then\ \int_A f \geq 0\ \ and\ if\ f \leq g\ \ ae\ then\ \int_A f \leq \int_A g$
    4. $if\ mA = 0\ \ and\ f\ is\ measurable\ then\ \int_A f = 0$

FUNCTIONS OF SEVERAL VARIABLES

We shall be concerned with functions of the type $f: D \to R$, where D is a subset of $R \times R = \{(x,y): x, y \in R$. We use the convention that D is the largest set for which $f: D \to R$ is a function. Throughout, $z = f(x,y)$ denotes a real valued function of two real variables x and y.

LIMIT OF A FUNCTION

We say that $f(x,y) \to L$ as $(x,y) \to (a,b)$ denoted by $\lim_{(x,y)\to(a,b)} f(x,y) = L$, if given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ for every $(x,y) \neq (a,b)$ satisfying $|x - a| < \delta$ and $|y - b| < \delta$.

RESULT

Suppose that $f(x,y) = \{\frac{p(x,y)}{q(x,y)},\ (x,y) \neq (0,0)\ c,\ (x,y) = 0 \geq 0$, p & q are polynomials in x and y with

$p(0,0) = q(0,0) = 0$, then,f has limit at $(0,0)$ $\Leftrightarrow deg\ deg\ p > deg\ deg\ q$.

EXAMPLES

- $f(x,y) = \{\frac{xy}{x^2+y^2},\ (x,y) \neq (0,0)\ 0,\ (x,y) = (0,0)$,
  here f has no limit at $(x,y) = (0,0)$, since, through
  $y = mx, f(x,y) = f(x,mx) = \frac{xmx}{x^2+m^2x^2} = \frac{m}{1+m^2},$
  depends on m, thus f is not continuous at $(0,0)$.
- $f(x,y) = \{\frac{xy(x^2-y^2)}{x^2+y^2},\ (x,y) \neq (0,0)\ 0,\ (x,y) = (0,0)$,
  here f has limit '0' at $(x,y) = (0,0)$, (through any curve $f \to 0$ as $(x,y) \to (0,0)$).

RESULT

$\lim_{y\to b} f(x,y) \neq \lim_{x\to a} f(x,y) \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

PARTIAL DERIVATIVES AND DIRECTIONAL DERIVATIVES

DEFINITION

- The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function with respect to the variable.
- Partial derivative of $f(x,y)$ with respect to x is generally denoted by $\frac{\partial f}{\partial x}$ or f_x or $f_x(x,y)$ while those with respect to y are denoted by $\frac{\partial f}{\partial y}$ or f_y or $f_y(x,y)$.

$$\frac{\partial f}{\partial x} = \frac{f(x+\delta x,\ y) - f(x,y)}{\delta x}$$

$$\frac{\partial f}{\partial y} = \frac{f(x,\ y+\delta y) - f(x,y)}{\delta y}$$

  When these limits exist.
- The partial derivatives at a particular point (a,b) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a,b)}, \frac{\partial f(a,b)}{\partial x} \quad or\ f_x(a,b)$$

$$\left[\frac{\partial f}{\partial y}\right]_{(a,b)}, \frac{\partial f(a,b)}{\partial y} \quad or\ f_y(a,b)$$

$$f_x(a,b) = \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \frac{f(a,b+k) - f(a,b)}{k}$$

  In case the limits exist.

RESULT

- Max/Min. value of function $f: R^2 \to R$
  (i) Find out critical points (a,b), from $f_x = 0, f_y = 0$.

(ii) Condition for the existence of extremum at (a, b) is

$$f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 > 0$$

(iii) $f_{xx}, f_{yy} > 0 \Rightarrow f(a,b)$ is min.

(iv) $f_{xx}, f_{yy} < 0 \Rightarrow f(a,b)$ is max.

(v) $f_{xx} < 0 \,\& f_{yy} > 0$ or $f_{xx} > 0 \,\& f_{yy} < 0 \Rightarrow$ test fails.

DIRECTIONAL DERIVATIVE IN THE DIRECTION OF A UNIT VECTOR $\hat{u}$

$D_{\hat{u}}f(a,b) = \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$ $\hat{u} = (u_1, u_2)$ is a unit vector.

NOTE

- $D_{(1,0)}f(a,b) = \frac{f(a+h,b)-f(a,b)}{h} = \frac{\partial f}{\partial x}(a,b)$
- $D_{(0,1)}f(a,b) = \frac{f(a,b+h)-f(a,b)}{h} = \frac{\delta f}{\delta y}(a,b)$

DIFFERENTIAL

Consider a function $z = f(x,y)$. Let us now give x an increment Δx and give y an increment Δy, and suppose that this result in an increment Δz for z, then $z + \Delta z = f(x + \Delta x, y + \Delta y)$ gives

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

SUFFICIENT CONDITION FOR DIFFERENTIABILITY

$\frac{\partial f}{\partial x} \,\&\, \frac{\partial f}{\partial y}$ exists and atleast one of them is continuous in some neighbourhood of $(a,b) \Rightarrow f$ is differentiable at (a,b).

LIMIT OF A FUNCTION $f: R^2 \to R^2$

- A function $f: R^2 \to R^2$can be given by $f(x,y) = \left(f_1(x,y), f_2(x,y)\right)$
- $f(x,y) = \left(\lim_{(x,y)\to(a,b)} f_1(x,y), \lim_{(x,y)\to(a,b)} f_2(x,y)\right)$

JACOBIAN MATRIX OF THE DERIVATIVE OF $f: R^2 \to R^2$

$$[Df_{(a,b)}](x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) & \frac{\partial f_2}{\partial x}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \end{pmatrix} (x\ y)$$

- f is linear $\Rightarrow Df = f$.

EXAMPLE

let $f(x,y) = (x, -y) \Rightarrow [Df_{(a,b)}](x,y) = (1(a,b)\ 0(a,b)\ 0(a,b)\ -1(a,b))(x,y) = (x, -y)$

RESULTS

- $f(x,y) = c$, a constant,$\forall(x,y) \in D \subseteq R^2 \Rightarrow Df = 0\forall(x,y) \in D$.
- Let $D \subseteq R^2, D$ is connected, $f(x,y) = c$, a constant, $\forall(x,y) \in D \Leftrightarrow Df = 0, \forall(x,y) \in D$.
- $f: R^2 \to R^2 \& \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) \Rightarrow f(x,y) - f(y,x) = (x-y)\left(\frac{\partial f}{\partial x}(x_0,y_0) - \frac{\partial f}{\partial y}(x_0,y_0)\right)$ for some $(x_0,y_0) \in R^2$.

MAXIMA AND MINIMA OF ONE VARIABLE

For a function of one variable, $f(x)$, we find the local maxima/minima by differentiation. Maxima/minima occur when $f'(x) = 0$.

- $x = a$ is a maximum if $f'(a) = 0$ and $f''(a) < 0$
- $x = a$ is a minimum if $f'(a) = 0$ and $f''(a) > 0$

A point where $f''(a) = 0$ and $f'''(a) \neq 0$ is called point of inflection.

Geometrically, the equation $y = f(x)$ represents a curve in the two dimensional (x,y) plane, and we call this curve the graph of the function $f(x)$.

MAXIMA AND MINIMA OF TWO VARIABLES

Our aim is to generalise these ideas to functions of two varibles. Such a function would be written as

$$z = f(x,y)$$

Where x and y are the independent variables and z is the dependent variable. The graph of such a function is a surface in three dimensional space. A simple example might be

$$z = \frac{1}{1 + x^2 + y^2}$$

z is the hight of the surface above a point (x,y) in the x-y plane. For functions $z = f(x,y)$ the graph (i.e, the surface) may have maximum points or minimum points (or both). But for surface there is a thrid possibility – a asddle point.

A point (a,b) which a maximum, minimum or saddle point is called a stationary point. The actual value at a stationary point is called the stationary value. What we need is a mathematical method for finding the stationary points of a function $f(x,y)$ and classifying them into maximum, minimum or saddle point. This method is analogous to, but more complicated than, the method of working out first and second derivatives for functions of one variable.

Let's remind ourselves about partial derivatives. The sort of function we have in mind might be something like

$$f(x,y) = x^2y^3 + 3y + x$$

And the partial derivatives of theis would be

$$\frac{\partial f}{\partial x} = 2xy^3 + 1$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 + 3$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2 = \frac{\partial^2 f}{\partial x \partial y}$$

Note that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This is true for any well behaved functions. In terms of notation, we will frequently use the other, subscript, notation for partial derivatives:

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y},$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \;\; or \;\; \frac{\partial^2 f}{\partial y \partial x}$$

FINDING STATIONARY POINTS

To find the stationary points of $f(x, y)$, work out $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and set both to zero. This gives you two equations for two unknowns x and y. Solve these equations for x and y (often there is more than one solution, as indeed you should expect. After all, even functions of one variable may have both maximum and minimum points).

CLASSIFYING STATIONARY POINTS

The procedure for classifying stationary points of a function of two variables is analogous to, but somewhat more involved, than the corresponding 'second derivative test' for functions of one variable. Below is, essentially, the second derivative test for functions of two variables:

Let (a, b) be a stationary point, so that $f_x = 0$ and $f_y = 0$ at (a, b). Then:

- If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) then (a, b) is a saddle point.
- If $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) then (a, b) is either a maximum or a minimum.
  Distinguish between these as follows:
  If $f_{xx} < 0$ and $f_{yy} < 0$ at (a, b) then (a, b) is a maximum point
  If $f_{xx} > 0$ and $f_{yy} > 0$ at (a, b) then (a, b) is a minimum point
- If $f_{xx} - f_{xy}^2 = 0$ then anything is possible.

INVERSE AND IMPLICIT FUNCTION THEOREMS

INVERSE FUNCTION

Suppose f is a one-to-one function with domain set A and range set B, then the inverse function f^{-1} has domain as B and range aa A and is defined as

$$f^{-1}(y) = x \ if \ and \ only \ if \ f(x) = y \ for \ any \ y \in B$$

IMPLICIT FUNCTION

An implicit function is a function of two independent variables, i.e. the functions of the form $f(x, y) = c$ are called implicit functions.

INVERSE FUNCTION THEOREM

Let D be an open subset of R^n and let $f: D \to R^n$ be a continuously differentiable function on D. Also, assume a base point $x_0 \in D$ and $f'(x_0): R^n \to R^n$ is invertible. Then there are open sets $U \subset D$ and $V \subset R^n$ containing $x_0 \ and \ f(x_0)$, respectively such that f is a bijection from $U \ to \ V$. hence there is an inverse map $f^{-1}: V \to U \ and \ f^{-1}$ is differentiable at y_0, then

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

IMPLICIT FUNCTION THEOREM

Let F be a continuous real valued function defined on some neighbourhood N of the fixed point (a, b) such that $F(a, b) = 0$ and there exists a continuous differentiable function $\frac{\partial F}{\partial y} \ on \ N$ where $\frac{\partial F}{\partial y}(a, b) \neq 0$. Then there is a unique function g defined on some neighbourhood N_a of a so that $g(a) = b, F(x, g(x)) = 0 \ for \ each \ x \in N_a$ and g is continuous.

Moreover, if $\frac{\partial F}{\partial x}$ exists and is continuous on N, then we can say that g is continuous differentiable function on N_a. Therefore, g' is given as

$$g'(t) = \frac{-\frac{\partial F}{\partial x}(t, g(t))}{\frac{\partial F}{\partial y}(t, g(t))} \quad , t \in N_a$$

GAMMA AND BETA FUNCTIONS

GAMMA FUNCTION

DEFINITION

We define Gamma function as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

RESULTS

- $\Gamma 1 = 1$
- $\Gamma \frac{1}{2} = \sqrt{\pi}$
- $\Gamma(n+1) = n\sqrt{n}$
- $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$

EXTENSION OF GAMMA FUNCTION FROM FACTORIAL NOTATION

- When n is a positive integer
  $\Gamma(n+1) = n\,(n-1)\,(n-2)\ldots 1 = n!.$
  Ex: $\Gamma 4 = 3!$
- When n is a positive rational number
  $\Gamma n = (n-1)(n-2)\ldots up\ to\ a\ positive\ number\ in\ \Gamma\ function.$
  Ex: $\Gamma \frac{7}{2} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2} = \frac{15\sqrt{\pi}}{8}$
- When n is a negative rational number
  Using $\Gamma(n+1) = n\,\Gamma n$

  $\Gamma n = \frac{\Gamma(n+1)}{n} = \frac{(n+1)\Gamma(n+1)}{n(n+1)}.$

  $= \frac{\Gamma(n+2)}{n(n+1)}.$

  $= \frac{\Gamma(n+3)}{n(n+1)(n+2)\ldots}.$

  Continuing this manner we get $\Gamma n = \frac{\Gamma(n+k+1)}{n(n+1)\ldots(n+k)}$ where k is the least positive integer such $that\ (n+k+1) > 0$

  Ex: $\Gamma(-3.4) = \frac{\Gamma(-3.4+k+1)}{(-3.4)(-2.4)\ldots(-3.4+k}, (-3.4 + k + 1) > 0$

  $\Rightarrow k > 2.4 \Rightarrow k = 3.$

  $\therefore \Gamma(-3.4) = \frac{\Gamma(-3.4+4)}{(-3.4)(-2.4)(-1.4)(-0.4)} = \frac{\Gamma 0.6}{(-3.4)(-2.4)(-1.4)(-0.4)}.$

  $\Gamma 0.6\ can\ be\ found\ using\ tables.$

- Γn is not defined when $n = 0$ or a negative integer

BETA FUNCTION

Beta function is defined as

ENTRI

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx, \qquad m,n>0$$

RESULTS

- Beta function is symmetric

$$\beta(m,n) = \beta(n,m)$$

- $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx,\ \ m,n>0$

- $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\,cos^{2n-1}\theta\,d\theta$

- $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

RELATION BETWEEN BETA AND GAMMA FUNCTIONS

The connection between the beta function and gamma function is given by

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}, \qquad m,n>0$$

RESULTS

- $\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}, m>0$

- $\Gamma n\Gamma(1-n) = \frac{\pi}{sinsin\,n\pi}, 0<n<1$