Many dynamical systems in Physics are ”non-linear”. These systems are described by equations (or differential equations) that do not have the properties of linear ones. For example, suppose $x_1(t)$ and $x_2(t)$ are possible functions for the position on the $x-axis$ for a system consisting of one particle. Let a and b be constants. If $(ax_1(t) + bx_2(t))$ is also a possible function for the position of the particle, then the one particle system is a linear one. If not, it is non-linear.

The properties of non-linear systems are quite different than linear ones. Some of these properties were discovered from a simple iterative mapping equation, the logistics map. The logistics map is also used as an introduction to the period-doubling route to chaos, and deterministic chaos. We start our treatment of nonlinear systems with this classic example.

The Logistic Map

Consider the function $f(x)$, which generates a series of numbers in the following manner:

$$x_{n+1} = f(x_n) \tag{1}$$

where n is an integer. Given an initial ”seed”, x_0, this equation generates a series of numbers.

This ”iterative map” approach is one used by ecologists in describing certain biological systems. For example, x_n could represent the number of fish in a pond for the n’th year. Suppose that the fish eggs hatch in spring, the fish grow in summer, the fish lay eggs in fall and then die. For this type of fish species, x_n represents the number of fish that hatch in spring. If conditions are good in year n, then perhaps $x_{n+1} > x_n$. Under bad conditions, $x_{n+1} < x_n$.

The simplest model for $f(x)$ is $f(x) = ax$ where a is a real constant. That is:

$$x_{n+1} = ax_n \tag{2}$$

If $a < 1$, then $x_n \to 0$ as $n \to \infty$. This situation might represent the case of a pond going ”bad” and the fish go extinct. If $a > 1$, then x_n gets larger each year by the factor a. This situation might represent the case when there are a small initial number of fish without any preditors. Of course, the fish population can’t keep increasing forever. This simple model, $f(x) = ax$, is a linear model for the fish pond, and is not generally representative of real populations in nature. To more accurately model a real population system, $f(x)$ needs to be nonlinear. A simple nonlinear model is to have $f(x) = ax - bx^2$, with a and b greater than zero. The linear term causes the population to increase, and the quadratic term causes the population to decrease. The iterative map is:

$$y_{n+1} = ay_n - by_n^2 \tag{3}$$

One can rescale y such that there is only one parameter for the map. If we let $y = (a/b)x$, then we have

$$\frac{a}{b}x_{n+1} = \frac{a^2}{b}x_n - \frac{a^2}{b}x_n^2 \tag{4}$$

Dividing by (a/b) yields

$$x_{n+1} = a(x_n - x_n^2) \tag{5}$$

In most textbooks, the constant a is named r. With this notation the iterative map takes the form:

$$x_{n+1} = rx_n(1 - x_n) \tag{6}$$

and is called the ”logistic map”, $f(r,x) = rx(1 - x)$. This simple equation yields many of the properties of non-linear systems. The allowed values for r are $0 \leq r \leq 4$. If r is restricted to this range, then the x_n will lie between zero and one: $0 \leq x_n \leq 1$.

Period Doubling Route to Chaos

We will demonstrate the period-doubling route to chaos with the logistic map $f(r,x_n)$. There is one parameter which characterizes the map, r, and is called the ”control parameter”. Once an initial seed x_0 is chosen then the series x_n, $n = 1 \to \infty$, can be generated for the particular control parameter r.

You will write a computer program to iterate the logistic map and examine the limiting values of x_n for large n. Here is what you should find:

1. For values of $0 \leq r \leq 1$, the series will decrease to a limiting value of $x_n \to 0$ as $n \to \infty$.

2. For values of $1 \leq r < 3$, the series will converge to a single value as $n \to \infty$. We will show that the value that the series converges to is $x_n \to (1 - 1/r)$ as $n \to \infty$.

You might wonder if these limiting values depends on the choice of the seed, x_0. It turns out that it does not. For any seed value $0 < x_0 < 1$ the limiting value is the same. We call this type of behavior ”Period-1”, because it takes one iteration for the x_n series to repeat in the limit of large n. In the fish pond analogy, the meaning is that the pond is a stable ecological system, with the same number of fish born every spring.

As r is increased, the limiting values of x_n show very interesting behavior. What you will find is that for $3 < r \leq\sim 3.449$, the x_n alternate between two values. There are two limiting values, which alternate between each other. This type of behavior is called ”Period-2”, since it takes two iterations for the x_n series to repeat (in the limit of large n). In the fish pond analogy, the pond would have a large number of fish one year, the next year a small number of fish, the next year would yield the large number of fish, etc. This sort of period-2 behaviour actually occurs in some biological systems. As with the period-1 case, the limiting values do not depend on the choice of the initial seed.

If you increase r past 3.449 you will notice that it takes 4 interations for the series to repeat. This ”period-4” behavior will continue until you reach $r \approx 3.544$. As you increase r past ≈ 3.544 you should see that it takes 8 interations for the series to repeat. This period-doubling behavior continues until the value of r is approximately 3.5699. For values of r just above 3.5699, say $r = 3.6$, the x_n series never repeats! This type of behavior is refered to as ”deterministic chaos”.

We summarize the critical values for r for which the period doubles in the table below. These values of r are transendental numbers, and we only list them to seve significant figures.

Value of r	Behavior
$r_1 = 3.0$	Period 2 begins
$r_2 \approx 3.449489$	Period 4 begins
$r_3 \approx 3.544090$	Period 8 begins
$r_4 \approx 3.564407$	Period 16 begins
$r_5 \approx 3.568759$	Period 32 begins
$r_6 \approx 3.569691$	Period 64 begins
$r_7 \approx 3.569891$	Period 128 begins
$\cdots$	…
r_N	Period 2^N begins
$\cdots$	…
$r \approx 3.569946$	Deterministic Chaos begins

You probably notice that r needs to be increased by smaller amounts for the next period doubling to occur. In fact there is a value for r_N less than ≈ 3.569891 for which the behavior is period-2^N for any N. Feigenbaum, a particle physicist, was the first to realize that there is a pattern for the period doubling: r needs to be increased by a factor of ≈ 1/4.6692 for the next period doubling. That is, if we define δ_N as:

$$\delta_N = \frac{r_N - r_{N-1}}{r_{N+1} - r_N} \tag{7}$$

Then, δ_N approaches a constant value as $N \to \infty$. That is,

$$lim_{N\to\infty}\frac{r_N - r_{N-1}}{r_{N+1} - r_N} \to 4.66920\ldots \tag{8}$$

This is a nice result. However, is it only true for the logistic map? Feigenbaum tried the iterative map

$$x_{n+1} = rsin(\pi x_n) \tag{9}$$

with control parameter r. As r is increased, the limiting values of x_n demonstrate period doubling similar to the logistic map. However, the values of r when period doubling occurs is different for this map than the logistic map. That is, the r_N are different for the two maps. He next evaluated the limit of

$$\delta_N = \frac{r_N - r_{N-1}}{r_{N+1} - r_N} \tag{10}$$

for $f(r,x_n) = rsin(\pi x_n)$. To Feigenbaum's amazement the limiting value of δ_N as $N \to \infty$ was also 4.66920... as with the logistic map! He discovered that there are a large number of maps which undergo the period-doubling route to chaos with the same limit for δ_N as $N \to \infty$. These types of maps are called quadratic maps. The period-doubling constant 4.66920... is often called Feigenbaum's constant.

Properties of Deterministic Chaos

If a system has its "control parameters" in the chaotic regime, the system can exhibit interesting properties. For the logistic map, this would be for certain ranges of r greater than 3.569891. We summarize some of the properties below:

1. There is no periodicity in the system. For iterative maps, the x_n do not have any repeating pattern. It is important to realize that the x_n are not random, they are

generated by a definate formula. Thus, the x_n series is deterministic. Hence the name deterministic chaos.

2. There is a great sensitivity to the initial conditions, i.e. the initial seed. Considertwo different initial seeds that are very close to each other, say x_0 and $y_0 = x_0$ + ^{2}. Let x_n be the n'th element in the series that started with x_0, and let y_n be the n'th element in the series that started with y_0. For systems in the chaotic regime, the x_n and y_n diverge exponentially from each other for small n no matter how small ^{2} is. For large n, the x_n and y_n are very different and one could never predict that they both started with nearly the same seed and are being generated by the same iterative map.

3. To predict the values of x_n for large n exactly, one needs to know x_0 to infinite accuracy. Due to the sensitivity to the initial conditions, any uncertainty in the initial starting value x_0 eventually become amplified until all predictability is lost.

In lecture, we will discuss the reasons for these properties.

Analysis of the Logistic Map

The period doubling route to chaos for quadratic maps have some universal properties. In particular, they all have the same "period doubling constant", or Feigenbaum constant. To understand why period doubling might occur we can analyze the logistic map. The logistic map $x_{n+1} = rx_n(1 - x_n)$ is simple and has the properties of quadratic maps in general.

For an iterative map to have periodicity one, or **period one**, means that as $n \to \infty$ the x_{n+1} become equal to x_n. For a particular value of r, one can solve the equation:

$$x_{n+1} = x_n = rx_n(1 - x_n) \tag{11}$$

Solving for x_n yields

$$\begin{aligned} x_n &= rx_n(1 - x_n) \\ x_n &= 1 - \frac{1}{r} \end{aligned}$$

So if x_n were to equal $(1 - 1/r)$, then the series would repeat this value of x forever. We will call this value of x the repeating value and label it x_R. This result is true for any r between one and four. Then, how does **period-2** behavior happen? The key is to examine the convergent/divergent properties of the x_n near the repeating value, i.e. near $x_R = (1 - 1/r)$.

Consider a value of x near the repeating value x_R, located a distance ^{2} away: $x_n = x_R + {}^2$. What is x_{n+1}?

$$x_{n+1} = f(r,x_n) = f(r,x_R + {}^2) \tag{12}$$

We can expand $f(r,x_R + {}^2)$ about x_R: $f(r,x_R + {}^2) = f(r,x_R) + {}^2(\partial f/\partial x)|_{x_R} + \cdots$. Substituting into the equation above gives:

$$\begin{aligned} x_{n+1} &= f(r,x_R) + \epsilon \frac{\partial f}{\partial x}|_{x_R} + \cdots \\ x_{n+1} &= x_R + \epsilon \frac{\partial f}{\partial x}|_{x_R} + \cdots \end{aligned}$$

From this equation, we see that if $|(\partial f/\partial x)|_{x_R}| < 1$ then x_{n+1} is closer to x_R than x_n was. In this case, as the series continues the terms will converge to x_R. However, if $|(\partial f/\partial x)|_{x_R}| > 1$ then x_{n+1} is further away from x_R than x_n was. In this case, as the series continues the terms will diverge from x_R. x_R will be an unstable repeating value. Let's calculate the value of r for which this occurs for the logistic map. We first need to calculate $(\partial f/\partial x)|_{x_R}$ for $f(r,x) = rx(1 - x) = rx - rx^2$:

$$\frac{\partial f}{\partial x} = r - 2rx \tag{13}$$

We now evaluate this expression at the repeating point $x = x_R = (1 - 1/r)$ and obtain:

$$\begin{aligned} \frac{\partial f}{\partial x}|_{x_R} &= r - 2rx_R \\ &= r - 2r(1 - \frac{1}{r}) \\ &= 2 - r \end{aligned}$$

Thus, we see that the absolute value of $\frac{\partial f}{\partial x}|_{x_R}$ is less than one for values of r between one and three. This is why, for $1 < r < 3$, $lim(n \to \infty)x_n$ is a repeating single value.

Period Doubling

For $r > 3$, the value $x = (1 - 1/r)$ is a repeating value. That is, if $x_0 = (1 - 1/r)$ so will all values of x_n. However, if x_0 is a little different than $(1 - 1/r)$ the x_n will not converge to a single repeating value. You might be worried that the series of x_n will diverge and never become stable. However, we have calculated the series and seen that if $3 < r < 3.449$ then x_n repeats after two iterations and is stable. This observation can assist in analyzing the map. We have demonstrated that the x_n are equal to each other after two iterations. That is,

$$x_{n+1} = f(r,x_n)$$
$$x_{n+2} = f(r,x_{n+1}))$$
$$x_{n+2} = f(r,f(r,x_n))$$

Suppose for $n \to \infty$ we have $x_{n+2} \to x_n$. The value of x_n for which this holds is found by solving for x in the "interated" equation:

$$x = f(r,f(r,x)) \tag{14}$$

We define $f^{(2)}(r,x)$ as

$$f^{(2)}(r,x) \equiv f(r,f(r,x)) \tag{15}$$

The "second iterate" function $f^{(2)}(r,x)$ can be analyzed using the same mathematical reasoning that we used for $f(r,x)$. That is, the stability of this "second iterate" of $f(r,x)$ will depend of the derivative of $f^{(2)}(r,x)$ evaluated at its repeating points. Let x_{R1} be a solution to the equation

$$x_{R1} = f^{(2)}(r,x_{R1}) \tag{16}$$

Note that there are two solutions for this "repeating point" of $f^{(2)}$. Let the other solution be x_{R2}:

$$x_{R2} = f^{(2)}(r,x_{R2}) \tag{17}$$

Where x_{R1} and x_{R2} satisfy

$$x_{R2} = f(r,x_{R1})\ x_{R1}$$
$$= f(r,x_{R2})$$

As discussed with $f(r,x)$, if the first derivative of $f^{(2)}$ evaluated at x_{R1} (or x_{R2}) lies between −1 and +1:

$$-1 < \frac{\partial f^{(2)}}{\partial x}|_{x_{R1}} < 1 \tag{18}$$

then x_{R1} is a stable repeating value for $f^{(2)}$. We can express the derivative of $f^{(2)}$ with respect to x in terms of $\partial f(r,x)/\partial x$ using the chain rule:

$$\frac{\partial f^{(2)}}{\partial x}|_{x_{R1}} = \frac{\partial f(r,x)}{\partial x}|_{x_{R2}} \frac{\partial f(r,x)}{\partial x}|_{x_{R1}} \tag{19}$$

For the logistic map and values of $3 < r < 3.449$ one of these derivatives has a magnitude greater than one and the other a value less than one. The magnitude of the product of the two derivatives is less than one, and the "period-two" behaviour is stable for r in this range. It is interesting to note that

$$\frac{\partial f^{(2)}}{\partial x}|_{x_{R2}} = \frac{\partial f(r,x)}{\partial x}|_{x_{R1}} \frac{\partial f(r,x)}{\partial x}|_{x_{R2}} = \frac{\partial f^{(2)}}{\partial x}|_{x_{R1}} \tag{20}$$

That is, each of the period two limiting values, x_{R1} and x_{R2}, have the same stability properties. At around $r \approx 3.449$, $\frac{\partial f^{(2)}}{\partial x}|_{x_{R1}}$ becomes equal to −1 and the repeating period two point becomes unstable. The instability also occurs at x_{R2} for $r \approx 3.449$. So, each of the two (period 2) values "bifurcate" at the same r value.

The next function to analyze is $f^{(4)}(r,x) = f^{(2)}(r,f^{(2)}(r,x))$, which is the fourth iterate of f. Now, there are four values of x that will satisfy

$$x_{R1} = f^{(4)}(r,x_{R1}) \tag{21}$$

As we did before, the stability of the solution to this equation will depend on the partial derivative of $f^{(4)}(r,x)$ with respect to x evaluated at x_{R1}. The same reasoning will apply to $f^{(4)}(r,x)$ as with $f(r,x)$ and $f^{(2)}(r,x)$, if the first derivative of $f^{(4)}$ evaluated at x_{R1} lies between −1 and +1:

$$-1 < \frac{\partial f^{(4)}}{\partial x}|_{x_{R1}} < 1 \tag{22}$$

then x_{R1} is a stable repeating value for $f^{(4)}$. We can express the derivative of $f^{(4)}$ with respect to x in terms of $\partial f(r,x)/\partial x$ using the chain rule:

$$\frac{\partial f^{(4)}}{\partial x}|_{x_{R1}} = \frac{\partial f(r,x)}{\partial x}|_{x_{R4}} \frac{\partial f(r,x)}{\partial x}|_{x_{R3}} \frac{\partial f(r,x)}{\partial x}|_{x_{R2}} \frac{\partial f(r,x)}{\partial x}|_{x_{R1}} \tag{23}$$

For the logistic map and values of $3.449 < r < 3.544$ this derivative lies between −1 and +1. However at $r \approx 3.544$ the left side of this equation equals −1 and the repeating values become unstable. Since each of the four repeating values have the same derivative for $f^{(4)}$, each of the four repeating values bifurcates at the same r value. This results in the periodicity going from 4 to 8.

Thus, we see that the periodicity goes from $1 \rightarrow 2 \rightarrow 4 \rightarrow 8$ as r is increased. Further increase of r yields a continual period doubling. The reason that the period doubles, as opposed to tripling, is due to the chain rule applied to the iterated functions.

Super Stable r values and Calculation of Feigenbaum Constant

One could calculate the values of r when period doubling occurs from the partial derivatives of $f(r,x)$. However, there is an easier way. We can take advantage of the way the partial derivatives change within the period doubling interval. Let the periodicity be labeled m, where $m = 2^n$. We have calculated the period 2 interval and found that $(\partial f^{(2)}(r,x)/\partial x)|_{x_R} = -1$ when the period 2 interval becomes unstable for x_R. For the next iterate, $f^{(4)}$, the start of the interval is at the same x_R, and the partial derivative is $\partial f^{(4)}(r,x)/\partial x)|_{x_R} = (\partial f^{(2)}(r,x)/\partial x)|_{x_R})(\partial f^{(2)}(r,x)/\partial x)|_{x_R}) = (-1)(-1) = 1$. Thus, the partial derivative of $f^{(4)}$ at the repeating points starts at +1. As r is increased the partial derivative at the repeating points decreases causing stability until the partial derivative equals −1. Then a bifurcation occurs. The process keeps repeating.

Within each period doubling interval of periodicity $m = 2^n$, the partial derivative of $f^{(m)}$ with respect to x begins with a value of +1 and ends with a value of −1 when the next bifurcation occurs. The reason for this is that $(-1)(-1) = +1$: $(\partial f^{(2m)}(r,x)/\partial x)|_{x_R} = (\partial f^{(m)}(r,x)/\partial x)|_{x_R})(\partial f^{(m)}(r,x)/\partial x)|_{x_R}) = (-1)(-1) = 1$. Therefore, somewhere near the middle of the interval the partial derivative of $f^{(m)}$ equals zero. The value of r for which the partial of $f^{(m)}$ equals zero is called the super-stable value, which we lable as s_m: $(\partial f^{(m)}(r,x)/\partial x)|_{s_m} = 0$. The repeating values have the most stability at this value of the control parameter r. We demonstrate this property for the period one interval.

As we discussed earlier, consider a value of x near the repeating value x_R, located a distance ^{2} away: $x_n = x_R + {}^2$. Then the next value in the series is x_{n+1} given by

$$x_{n+1} = f(r,x_n) = f(r,x_R + {}^2) \tag{24}$$

As before, we can expand $f(r,x_R+{}^2)$ about x_R: $f(r,x_R+{}^2) = f(r,x_R)+{}^2(\partial f/\partial x)|_{x_R}+ \cdots$. Substituting into the equation above gives:

$$\begin{aligned} x_{n+1} &= f(r,x_R) + \epsilon \frac{\partial f}{\partial x}|_{x_R} + \cdots \\ x_{n+1} &= x_R + \epsilon \frac{\partial f}{\partial x}|_{x_R} + \cdots \end{aligned}$$

If $\frac{\partial f}{\partial x}|_{x_R}$ equals zero, then x_{n+1} equals the repeating point. The convergence to x_R occurs very quickly in the series. Thus, the name super-stable. For the period one interval, s_1 can be found by solving

$$\begin{aligned} \frac{\partial f}{\partial x}|_{x_R} &= r - 2rx_R \\ &= r - 2r(1-\frac{1}{r}) \\ &= 2 - r \\ 0 &= 2 - r \\ r &= 2 \end{aligned}$$

At $r = 2$ the first derivative of $f(r,x)$ is zero at $x = 1-1/r$. There is an easier way to find the super stable value for r. Since $f(r,x) = rx(1-x)$, f **is symmetric about** $x = 1/2$ **for any value of** r. Therefore $\frac{\partial f}{\partial x}|_{x=1/2} = 0$ for any value of r. We can find the super-stable value of r in the period one interval without taking the partial derivative. We just need to find the value of r such that $f(r,1/2) = 1/2$:

$$\begin{aligned} \frac{1}{2} &= r\,\frac{1}{2}(1-\frac{1}{2}) \\ r &= 2 \\ s_1 &= 2 \end{aligned}$$

as before. The same reasoning can be applied to the period two interval. The first derivative of $f^{(2)}(r,x)$ equals

$$\frac{\partial f^{(2)}}{\partial x}|_{x_{R1}} = \frac{\partial f(r,x)}{\partial x}|_{x_{R2}} \frac{\partial f(r,x)}{\partial x}|_{x_{R1}} \quad (25)$$

using the chain rule. For this derivative to be equal to zero, one of the two derivatives of $f(r,x)$ must be zero. However, the derivative of $f(r,x)$ is only zero at $x = 1/2$. Therefore, **at the super stable value for r in the period 2 region, $r = s_2$, one of the values of x, x_{R1} or x_{R2}, must be** 1/2. Therefore we can find s_2 by solving the the equation:

$$\frac{1}{2} = f^{(2)}(s_2, \frac{1}{2}) \tag{26}$$

We apply the same reasoning to the period 4, 8, ⋯, $m = 2^n$ intervals. In order for the partial derivative of $f^{(m)}$ to be zero, one of the derivatives of $f(r,x_{Ri})$ must be zero. The zero partial derivative will only occur at $x = 1/2$. So, we can generalize:

$$\frac{1}{2} = f^{(m)}(s_m, \frac{1}{2}) \tag{27}$$

where $m = 2^n$ for the period 2^n interval. This equation can be solved for any period 2^n interval. Determining the super-stable values s_m is easier than solving for the bifurcation values of r, r_m. The super-stable values can be obtained by solving the single algebraic equation above. For the bifurcation values, one needs to solve two coupled equations: one setting the derivative to −1, and another solving for the repeating value.

By determining the super-stable values, one can get a good approximation for Feigenbaum's constant.

$$\delta = lim_{m\to\infty} \frac{s_{2m} - s_m}{s_{4m} - s_{2m}} \tag{28}$$

This is the method Feigenbaum used to estimate his constant.