Nonlinear Oscillation

Up until now, we've been considering the differential equation for the (damped) harmonic oscillator,

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = L_\beta y = f(t). \tag{1}$$

Due to the linearity of the differential operator on the left side of our equation, we were able to make use of a large number of theorems in finding the solution to this equation. In fact, in the last few lectures, we've pretty much been able to solve this equation for any realistic case we could imagine.

However, we know that this equation came from an approximation - we've been assuming that the potential energy function of the spring can be written as

$$U(y) = \frac{1}{2}ky^2. \tag{2}$$

While in many cases this is an incredibly good approximation, we may wonder how the the addition of higher order terms might affect the behaviour of our system. For example, we could imagine taking one more term in a hypothetical Taylor series expansion which defines $U(y)$, so that

$$U(y) = \frac{1}{2}ky^2 + \frac{1}{6}\gamma y^3. \tag{3}$$

A plot of this potential energy function is shown in Figure 1.

This is not, however, a particularly nice potential energy function to work with, because it is **unstable**. Because the cubic term we have added is an odd function, then for large negative values of y, assuming that $\gamma > 0$, we find

$$U(y \to -\infty) = -\infty, \tag{4}$$

instead of positive infinity. For the case that $\gamma < 0$, we find similar behaviour as we move in the opposite direction. The implication of this is that any particle subject to this potential energy function with a total energy larger than

$$E_c = \frac{2}{3}\frac{k^3}{\gamma^2} \tag{5}$$

will escape towards negative infinity, and never come back. This is also shown in Figure 1. While this may indeed describe some physical systems, it does not do a good job of modelling the type of system we are interested in, which is reasonably small oscillations around an equilibrium point.

For this reason, we should add at least one more term to the Taylor series expansion of the potential,

$$U(y) = \frac{1}{2}ky^2 + \frac{1}{6}\gamma y^3 + \frac{1}{24}\lambda y^4. \tag{6}$$

For $\lambda > 0$, this now describes a stable potential energy function. A plot of this improved potential energy expansion is shown in Figure 2. Notice that despite

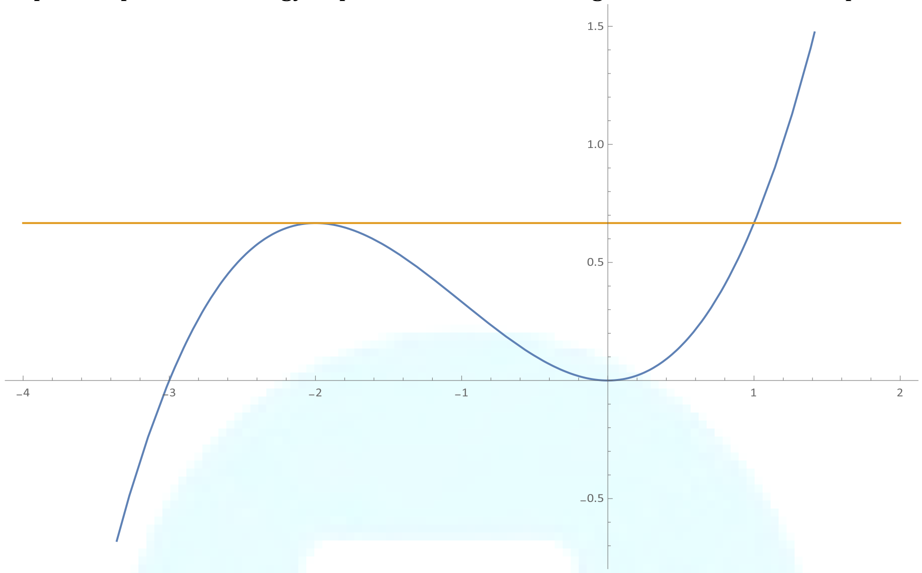

Figure 1: A plot of the unstable potential energy function (blue curve), for $k = \gamma = 1$. The orange line indicates the amount of energy a particle would need in order to be able to "hop" out of the potential minimum and travel towards negative infinity.

the somewhat strange shape near the origin, a particle with an arbitrarily large energy will be trapped inside of the potential minimum. Thus, we can consider motion at an arbitrarily large energy, without worrying about issues of stability. With this potential energy function, my differential equation now becomes

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y + \phi y^2 + \epsilon y^3 = f(t), \tag{7}$$

where

$$\phi = \gamma/2m \ ; \ \epsilon = \lambda/6m. \tag{8}$$

For simplicity, we will assume that there is no damping, no driving force, and that the cubic term in the potential is zero (so that the potential energy is symmetric around zero). In this case, we find the **Duffing equation**,

$$\ddot{y} + \omega^2 y + \epsilon y^3 = 0. \tag{9}$$

It is a nonlinear differential equation that describes a simple harmonic oscillator with an additional correction to its potential energy function. This type of oscillator is often known as an **anharmonic oscillator**. How do we solve for the motion of such a system?

The most important thing to notice before embarking upon our search for a solution is that in the presence of a nonlinear term, *the principle of superposition no longer applies*. That is, if I have two solutions to my differential

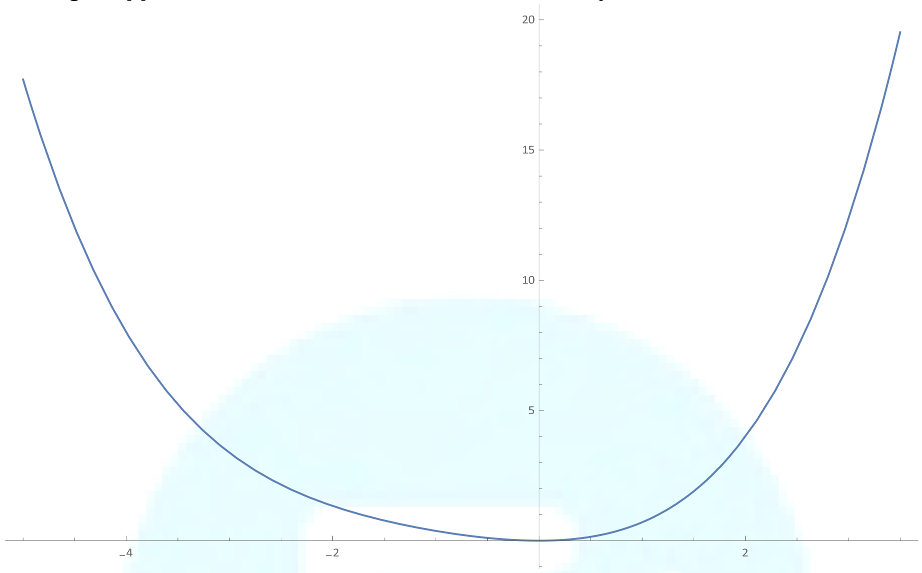

Figure 2: A plot of the stable potential energy function, for $k = \gamma = \lambda = 1$.

equation, $y_1(t)$ and $y_2(t)$, then a linear combination of these two solutions is NOT necessarily a solution, because, of course,

$$(y_1 + y_2)^3 \neq y_1^3 + y_2^3. \tag{10}$$

Almost every solution technique we have used so far has, at least in some way, involved the principle of superposition, a property which we have now lost. So what now?

While we may no longer be able to use the principle of superposition, we do have one old tool which we can always fall back on: perturbation theory. Our goal here is to understand how, under a suitable approximation, we can think of the motion of the anharmonic oscillator as being a "perturbation" of the harmonic oscillator's motion. For nonlinear problems, there will often be many different ways to perform perturbation theory, each with their advantages and disadvantages. We'll explore two techniques here, although this list is far from being exhaustive.

Small-Parameter Perturbation Theory

Let's imagine that the quantity describing the anharmonic term in our potential is sufficiently "small." Actually, more carefully, we should say that

$$\epsilon y^3 \ll \omega^2 y \tag{11}$$

for the entire region in which the particle moves (can you see why the cubic term will always become important for large enough *y*, no matter how small is?). While we may not know how to solve for the motion of the particle exactly, we do know how to find the region in which it travels, and thus we can check whether this condition holds. If the particle has total energy *E*, then its turning points must be the locations at which

$$U(y_M) = E \ \Rightarrow\ \frac{1}{2}ky_M^2 + \frac{1}{24}\lambda y_M^4 = E, \tag{12}$$

or,

$$y_M = \pm\sqrt{\frac{6k}{\lambda}}\sqrt{-1+\sqrt{1+2\lambda E/3k^2}}. \tag{13}$$

If the quartic part of the potential is indeed only a small correction in between these two points, then the nonlinear term in our differential equation should only represent a small perturbation to the linear oscillator, described by the differential equation,

$$\ddot{y} + \omega^2 y = 0, \tag{14}$$

whose solution we know to be

$$y_0(t) = A\cos(\omega t) + B\sin(\omega t). \tag{15}$$

Motivated by this thinking, we might imagine that in some sense, the solution to the anharmonic oscillator is given by a small "correction" to the harmonic solution, a correction which depends on the small quantity . Such a correction might look something like

$$y(t) = y_0(t) + \epsilon y_1(t) + \ldots \tag{16}$$

The function $y_1(t)$ is the correction term, and we can think of it as the next term in an expansion in powers of . In this case, however, notice that the coefficients in the expansion are *functions of time*. That is not unlike the previous cases we considered, where, for example, the expansion was in powers of the drag parameter, but the coefficients in this expansion could depend on the other parameters of the problem (mass, initial velocity, etc.) in a more complicated way. If we plug this proposed solution into our full differential equation, we find

$$\ddot{y}_0 + \omega^2 y_0 + \epsilon\ddot{y}_1 + \epsilon\omega^2 y_1 + \epsilon(y_0 + \epsilon y_1)^3 = 0. \tag{17}$$

If we expand this equation out to first order in epsilon, we find

$$\left(\ddot{y}_0 + \omega^2 y_0\right) + \epsilon\left(\ddot{y}_1 + \omega^2 y_1 + y_0^3\right) = 0. \tag{18}$$

Now, in order for both sides of this equation to be equal for all values of , it must be the case that each parenthetical term vanishes. In this case, the zero-order parenthetical term yields

$$\ddot{y}_0 + \omega^2 y_0 = 0, \tag{19}$$

which is nothing other than the equation for the linear oscillator, which we know how to solve. If we also match the first-order parenthetical term, then we have

$$\ddot{y}_1 + \omega^2 y_1 = -y_0^3. \tag{20}$$

Since we already know what y_0 is, this represents a forced, *linear* differential equation for y_1, which is something we do know how to solve. In particular, this equation describes the function y_1 as the coordinate of a simple harmonic oscillator with frequency ω.

For concreteness, let's assume we've chosen our initial conditions such that

$$y\,(t = 0) = 1\ ;\ v\,(t = 0) = 0, \tag{21}$$

where I've avoided using y_0 and v_0, so as to not confuse them with the coefficients in the expansion. Applying these initial conditions to the zero order solution, we have

$$y_0\,(t) = \cos(\omega t). \tag{22}$$

Our perturbative equation then tells us

$$\ddot{y}_1 + \omega^2 y_1 = -\cos^3(\omega t), \tag{23}$$

Using a trigonometric identity, we can rewrite the cubed term as

$$\ddot{y}_1 + \omega^2 y_1 = -\frac{3}{4}\cos(\omega t) - \frac{1}{4}\cos(3\omega t). \tag{24}$$

This is just a linear, undamped oscillator subject to two sinusoidal driving forces. As for its solution, we can simply quote our results from the past lecture, in order to find

$$y_1\,(t) = A\cos(\omega t) + \left(B - \frac{3}{8\omega}t\right)\sin(\omega t) + \frac{1}{32\omega^2}\cos(3\omega t), \tag{25}$$

where the constants A and B arise from the homogeneous part of the solution. Thus, the full motion, through order , is given by

$$y\,(t) = (A\epsilon + 1)\cos(\omega t) + \epsilon\left(B - \frac{3}{8\omega}t\right)\sin(\omega t) + \frac{\epsilon}{32\omega^2}\cos(3\omega t). \tag{26}$$

If we apply the same initial conditions as before, a short calculation reveals

$$A = -\frac{1}{32\omega^2}\ ;\ B = 0, \tag{27}$$

and so we find

$$y\,(t) = \left(1 - \frac{\epsilon}{32\omega^2}\right)\cos(\omega t) + \frac{\epsilon}{32\omega^2}\cos(3\omega t) - \frac{3\epsilon}{8\omega}t\sin(\omega t). \tag{28}$$

Notice that the coefficient on $\cos(\omega t)$ is slightly less than one, by the same quantity which multiples the $\cos(3\omega t)$ term. Thus, some of the “weight” of the solution has been transferred into an oscillatory term with a frequency that is an integer multiple of the original frequency. We say that the nonlinear “interaction” has “excited a higher harmonic” of the oscillator, which is a general feature of nonlinear differential equations. While the linear system required an external driving force in order to excite higher harmonics, the nonlinear system is capable of doing so under the action of its own internal dynamics. This type of behaviour would also appear, for example, in a course on the Standard Model, in which a similar type of differential equation, this time describing something known as a Quantum Field, would be used to describe how the interaction of particles can create and destroy new particles.

The Poincar´e-Lindstedt Method

There is, however, an obvious problem with the answer we have found from using perturbation theory. Notice that the sine term has a factor of t - it continues to grow over time, increasing without bound. This is totally inconsistent with the behaviour we expect on physical grounds - the particle should simply oscillate back and forth between the two turning points. The appearance of this term is actually quite general, and not special to this case. The term appears because in the expansion of y_0^n, for odd values of n, there will always be a sinusoidal term with the undamped frequency of the linear oscillator,

$$\cos^n(\omega t) = \alpha\cos(\omega t) + ... \tag{29}$$

where

$$\alpha = \frac{2}{2^n}\binom{n}{\frac{n-1}{2}} \tag{30}$$

involves a binomial coefficient. Even powers also cause problems, although this only becomes clear at higher orders in perturbation theory. Because the differential equation for y_1 has the same natural frequency ω as the linear oscillator,

$$\ddot{y}_1 + \omega^2 y_1 = -y_0^3, \tag{31}$$

then there is an undamped resonance, resulting in this diverging oscillation amplitude. This type of term, one which arises in perturbation theory and grows without bound over time, is often known as a **secular term.**

The resolution to this problem comes from realizing a second, somewhat more subtle problem with our solution - it oscillates at the wrong frequency. The frequencies which appear in our solution are ω, and also a harmonic multiple 3ω. Thus, our solution is still periodic with frequency ω. However, this is inconsistent with the fact that in addition to changing the functional form of the solution, the quartic perturbation to the potential will in general **also** change the oscillation frequency of the spring. In some sense, the reason that our naive version of perturbation theory has failed is because we have not taken into account the fact that the new frequency of oscillation in our system will no longer be the same as the frequency of the linear oscillator. So how do we take this fact into account?

As a fist step, notice that if y_0 were to oscillate at a frequency other than ω, we would no longer have our secular term problem - the forcing function would no longer be in resonance with the differential equation for y_1. This realization gives us the idea that maybe we can make a slightly better choice for our "unperturbed" solution, and instead choose

$$y_0(t) = \cos(\Omega t), \tag{32}$$

for some Ω 6= ω. This revised choice reflects an attempt to incorporate the additional change in the frequency of the oscillator, while still "perturbing" away from the solution we already know. So how should we choose Ω? One guess might be to simply replace it with the actual frequency of oscillation in the full potential, which we can find from the period,

$$T = \sqrt{2m} \int_{y_-}^{y_+} \frac{dx}{\sqrt{E - U(x)}}, \tag{33}$$

where y_- and y_+ are the two turning points. However, while this will give the correct oscillation frequency, perhaps it may seem as though our unperturbed solution y_0 should not necessarily incorporate the exact frequency of the oscillator - it is, after all, only supposed to be approximately correct, not exactly correct. Of course, we can always perform a perturbative calculation of Ω, writing the new frequency in terms of an expansion in ,

$$\Omega = \Omega_0 + \epsilon\Omega_1 + \ldots \tag{34}$$

as you did on the homework. However, it's not clear exactly how many terms we should take.

In order to side-step this thorny issue altogether, we will make use of a new tool, sometimes known as a dual series expansion. The idea is that we have two objects, the frequency Ω and the trajectory $y(t)$, which both need to be expanded in terms of . By expanding them simultaneously in just the right way, we can eliminate the secular term from our solution. In order to do so, we will in fact not modify the solution y_0 directly, but instead define a new time variable

$$\tau = \Omega t, \tag{35}$$

so that in terms of this new variable, our differential equation becomes

$$\Omega^2 \ddot{y}(\tau) + \omega^2 y(\tau) + \epsilon y^3(\tau) = 0. \tag{36}$$

The derivatives in the first term are now derivatives with respect to τ, and so the chain rule pulls out a factor of Ω^2, since

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \Omega\frac{d}{d\tau}. \quad (37)$$

Make sure to understand that this is exactly the same equation as before, simply written in terms of a new coordinate. The difference will come when we conclude later that Ω is something other than ω.

We now seek a solution of the form

$$y(\tau) = y_0(\tau) + \epsilon y_1(\tau) + \ldots \quad (38)$$

If we plug in this proposed solution, along with the expansion for Ω, we arrive at the equation

$$(\omega + \epsilon\Omega_1)^2 \ddot{y}_0 + \omega^2 y_0 + \epsilon(\omega + \epsilon\Omega_1)^2 \ddot{y}_1 + \epsilon\omega^2 y_1 + \epsilon(y_0 + \epsilon y_1)^3 = 0, \quad (39)$$

where we used the fact that

$$\Omega_0 = \omega, \quad (40)$$

which must be true in order to recover the correct value for the frequency when = 0. If we expand in and only keep terms up to first order, we find

$$\omega^2(\ddot{y}_0 + y_0) + \epsilon\left(2\omega\Omega_1\ddot{y}_0 + \omega^2\ddot{y}_1 + \omega^2 y_1 + y_0^3\right) = 0. \quad (41)$$

The vanishing of the first term tells us that

$$\ddot{y}_0 + y_0 = 0, \quad (42)$$

which has the general solution

$$y_0(\tau) = A\cos(\tau) + B\sin(\tau) = A\cos(\Omega t) + B\sin(\Omega t), \quad (43)$$

or, to zero order in the frequency,

$$y_0(\tau) = A\cos(\Omega_0 t) + B\sin(\Omega_0 t) = A\cos(\omega t) + B\sin(\omega t). \quad (44)$$

This is indeed the correct solution when $\epsilon = 0$.

So far, it would not appear that our new technique has accomplished anything. However, things start to look different when we consider the first order term,

$$\ddot{y}_1 + y_1 = -2\frac{\Omega_1}{\omega}\ddot{y}_0 - \frac{1}{\omega^2}y_0^3 \quad (45)$$

This is again a linear differential equation describing y_1, with a forcing term that depends on y_0, although it has a slightly different appearance. Again, choosing our initial conditions such that

$$y(t = 0) = 1\ ;\ \dot{y}(t = 0) = 0, \quad (46)$$

we find, according to the chain rule

$$y\,(\tau = 0) = 1\ ;\ \Omega\dot{}y\,(\tau = 0) = 0 \Rightarrow y\dot{}\,(\tau = 0) = 0, \tag{47}$$

so that our zero order solution is

$$y_0\,(\tau) = \cos(\tau). \tag{48}$$

Therefore, our first order equation reads

$$\ddot{y}_1 + y_1 = 2\frac{\omega_1}{\omega}\cos(\tau) - \frac{1}{\omega^2}\cos^3(\tau), \tag{49}$$

or, using the same trigonometric identity for the cosine cubed term,

$$\ddot{y}_1 + y_1 = \frac{2}{\omega}\left(\Omega_1 - \frac{3}{8\omega}\right)\cos(\tau) - \frac{1}{4\omega^2}\cos(3\tau). \tag{50}$$

Again, we find an equation for y_1 which contains an undamped resonance - the natural frequency of y_1 (in this case simply 1) is matched on the right by a sinusoidal forcing term with the same frequency. This resonance would cause y_1 (τ) to contain an overall factor of τ, which also goes to infinity for very large times. Thus, it would seem that we still have the same problem as before. However, notice that in this case, the resonant forcing term is multiplied by a factor which involves the expansion coefficient Ω_1. If we were to set

$$\Omega_1 - \frac{3}{8\omega} = 0, \tag{51}$$

then this problematic term would be gone, and we would have a well-behaved solution. Thus, we see how our secular term can actually be turned into a useful tool, rather than a problem. If I require that my solution be free of any problematic divergent terms (which I know must be the case), then this forces me to make a specific choice for ω_1, which helps me determine the series expansion for Ω. This technique is known as the **Poincar´e-Lindstedt Method**, and it is a very useful tool for studying periodic potion in a nonlinear potential. It can be continued to higher orders in , and at each step in the expansion, the elimination of a resonant forcing function will fix another term in the expansion of Ω.

Having made this choice for Ω_1, we find

$$\ddot{y}_1 + y_1 = -\frac{1}{4\omega^2}\cos(3\tau). \tag{52}$$

Quoting our result from before, the solution to this equation is

$$y_1\,(\tau) = \frac{1}{32\omega^2}\cos(3\tau) + A\cos(\tau) + B\sin(\tau), \tag{53}$$

where the last two terms come from the addition of the homogeneous solution. Therefore, our full solution, valid to first order in , is given by

$$y(\tau) = (A\epsilon + 1)\cos(\tau) + \frac{\epsilon}{32\omega^2}\cos(3\tau) + \epsilon B \sin(\tau). \quad (54)$$

If we fix the same boundary conditions as before, we find yet again

$$A = -\frac{1}{32\omega^2} \; ; \; B = 0, \quad (55)$$

Thus, our solution, in terms of τ, is given by

$$y(\tau) = \left(1 - \frac{\epsilon}{32\omega^2}\right)\cos(\tau) + \frac{\epsilon}{32\omega^2}\cos(3\tau). \quad (56)$$

Using our expansion for the frequency, this becomes, in terms of Ω and t,

$$y(t) = \left(1 - \frac{\epsilon}{32\omega^2}\right)\cos\left((\Omega_0 + \epsilon\Omega_1)\, t\right) + \frac{\epsilon}{32\omega^2}\cos\left(3\left(\Omega_0 + \epsilon\Omega_1\right) t\right), \quad (57)$$

where

$$\Omega_0 = \omega \; ; \; \Omega^1 = \frac{3}{8\omega}. \quad (58)$$

We now have a solution to our equation which is **stable** for all times - it has the correct oscillatory behaviour, and does not diverge to infinity. Also, notice that not only are there still higher harmonics which have been excited by the nonlinear interaction, the value of the base frequency has also been modified by the nonlinear interaction. The interplay of these two effects leads to a modified solution in the presence of the quartic perturbation. Figure 3 shows a plot of this solution, for three different values of . Notice that for small enough values of , the overall qualitative shape of the plot is the same, although the period of oscillation is slightly shorter, and the shape is not quite a pure sinusoidal term. Figure 4 shows a zoomed-in version of these three solutions, as they approach their initial starting values, demonstrating that the effect of increasing is to bring the oscillator back to its initial position sooner. Figure 5 compares the solution to a pure cosine term with the same amplitude and frequency - notice that the shape of the curve is slightly different as a result of the higher harmonic that has been introduced.

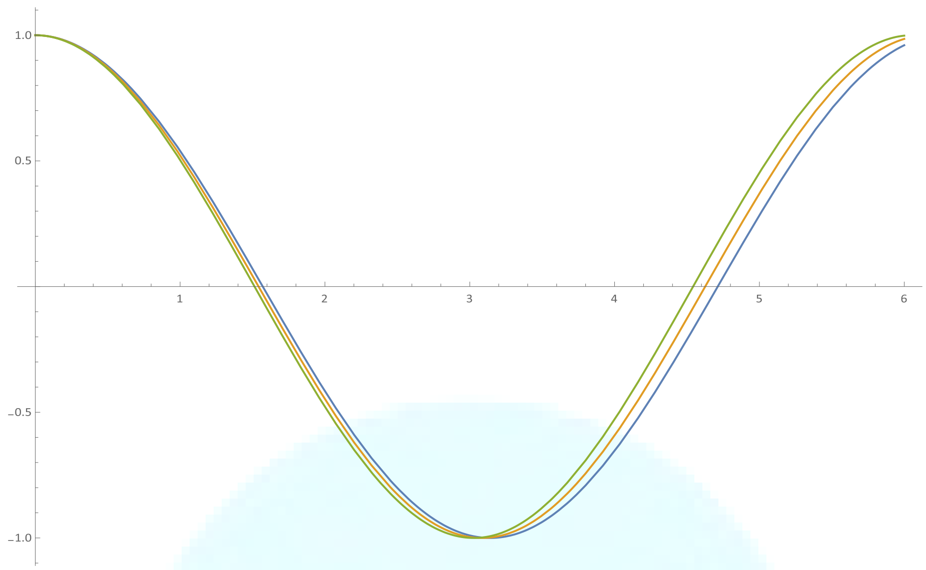

Figure 3: A plot of the perturbative solution to the nonlinear oscillator, for ω = 1 and = 0 (blue curve), $\epsilon = 0.05$ (orange curve), and $\epsilon = 0.1$ (green curve). Notice that all three solutions have qualitatively the same shape, and that the period decreases with increasing .

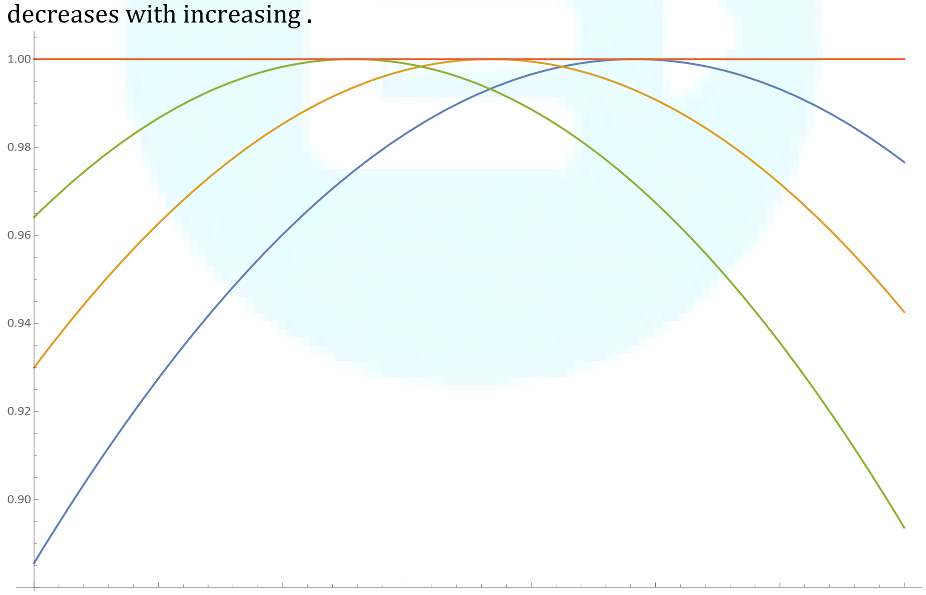

Figure 4: A plot of the perturbative solution to the nonlinear oscillator, for ω = 1 and = 0 (blue curve), $\epsilon = 0.05$ (orange curve), and $\epsilon = 0.1$ (green curve). The red line (at a value of 1) shows the amplitude of all three oscillators, clearly

demonstrating that the oscillators with larger return to their initial starting position sooner.

Short Time Perturbation Theory

For small enough values of , the above perturbative approach gives a nicely behaved approximate solution to our differential equation. However, there may be situations in which the quartic term is not small, and in fact may be a more important contribution than the quadratic term. In such a situation, our perturbative method above will not yield a solution which is reasonably accurate, and we must develop a new technique.

However, this does not mean that we are completely out of luck - there is of course another parameter in our problem which we can use to perform an expansion, the time coordinate itself. If we make the assumption that $y(t)$ admits a Taylor series expansion, so that

$$y(t) = \sum_{n=0}^{\infty} y_n t^n = y_0 + y_1 t + y_2 t^2 + \ldots \qquad (59)$$

then in principle, we can plug this expansion into our differential equation, and attempt to solve for the coefficients y_n. In particular, we have

$$\dot{y}(t) = \sum_{n=0}^{\infty} n y_n t^{n-1} = \sum_{n=1}^{\infty} n y_n t^{n-1}. \qquad (60)$$

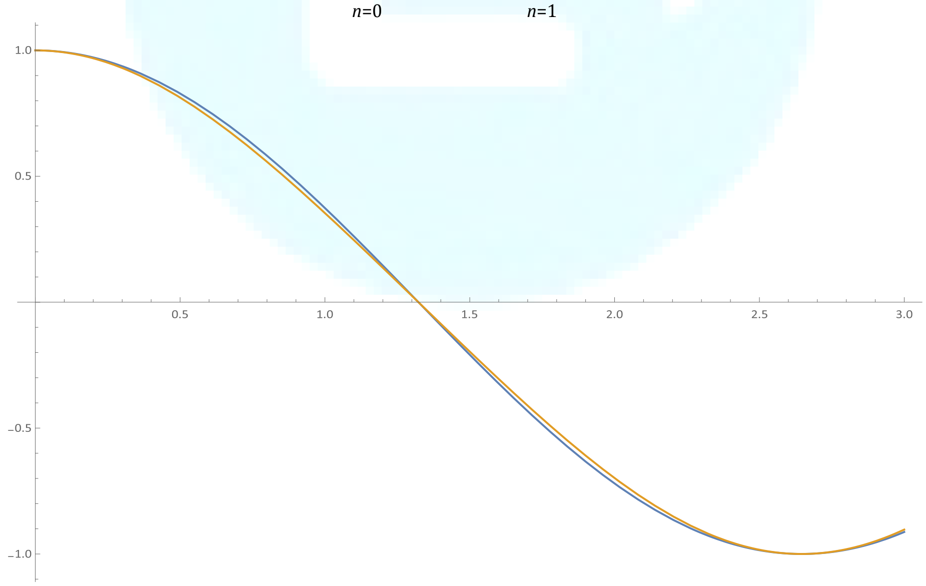

Figure 5: A plot of the perturbative solution to the nonlinear oscillator, for $\omega = 1$ and $\epsilon = 0.5$ (orange curve). The blue curve shows a pure cosine term with the same amplitude and frequency as the nonlinear oscillator. Notice that the presence of the higher harmonic changes the shape of the solution slightly away from a pure sinusoidal function.

Notice that the second sum can equally be taken to start at $n = 1$, since the $n = 0$ term is simply zero anyway. If we rewrite the summation index slightly,

$$m = n - 1, \tag{61}$$

then this summation becomes

$$\dot{y}(t) = \sum_{m=0}^{\infty} (m+1) y_{m+1} t^m. \tag{62}$$

Similarly, we can write

$$\ddot{y}(t) = \sum_{m=0}^{\infty} (m+1)(m+2) y_{m+2} t^m . \tag{63}$$

This then results in the differential equation

$$\left(\sum_{n=0}^{\infty} (n+1)(n+2) y_{n+2} t^n\right) + \omega^2 \left(\sum_{n=0}^{\infty} y_n t^n\right) + \epsilon \left(\sum_{n=0}^{\infty} y_n t^n\right)^3 = 0. \tag{64}$$

Now, in order for this to hold for all times t, it must again be true that both sides of the equation are equal, power by power in t. So our goal again is to expand the left side in powers of t, and match coefficients. Performing this expansion to second order, we find

$$2y_2 + 6y_3 t + 12 y_4 t^2) + \omega^2 \left(y_0 + y_1 t + y_2 t^2\right) + \epsilon \left(y_0^3 + 3y_0^2 y_1 t + 3y_0^2 y_2 t^2 + 3y_0 y_1^2 t^2\right) = 0 \tag{65}$$

Matching the zero order terms, we find the equation,

$$2y_2 + \omega^2 y_0 + \epsilon y_0^3 = 0, \tag{66}$$

or

$$y_2 = -\frac{\omega^2}{2} y_0 - \frac{\epsilon}{2} y_0^3. \tag{67}$$

This equation fixes y_2 in terms of y_0. However, we in fact already know y_0, since it is none other than the initial condition

$$y_0 = y(t = 0). \tag{68}$$

So assuming we know the initial position, we have now fixed the second order coefficient in the expansion.

Continuing on to match the first order term, we find

$$6y_3 + \omega^2 y_1 + 3\epsilon y_0^2 y_1 = 0, \tag{69}$$

or

$$y_3 = -\frac{1}{6}\omega^2 y_1 - \frac{1}{2}\epsilon y_0^2 y_1. \tag{70}$$

This fixes y_3 in terms of y_0 and y_1. We also have knowledge of y_1, however, since our initial condition for the velocity reads

$$v_0 = \dot{y}(0) = y_1. \tag{71}$$

Thus, we have

$$y_3 = -\frac{1}{6}\omega^2 v_0 - \frac{1}{2}\epsilon y_0^2 v_0. \tag{72}$$

Lastly, matching the second order term, we find

$$12y_4 + \omega^2 y_2 + 3\epsilon\left(y_0^2 y_2 + y_0 y_1^2\right) = 0, \tag{73}$$

or

$$y_4 = -\frac{1}{12}\omega^2 y_2 - \frac{\epsilon}{4}\left(y_0^2 y_2 + y_0 y_1^2\right). \tag{74}$$

Since we know all of the values on the right side, we can write this as

$$y_4 = -\frac{1}{12}\omega^2\left(-\frac{\omega^2}{2}y_0 - \frac{\epsilon}{2}y_0^3\right) - \frac{\epsilon}{4}\left(y_0^2\left(-\frac{\omega^2}{2}y_0 - \frac{\epsilon}{2}y_0^3\right) + y_0 v_0^2\right), \tag{75}$$

or, simplifying this a bit,

$$y_4 = \frac{\omega^4}{24}y_0 + \frac{\epsilon\omega^2}{6}y_0^3 + \frac{\epsilon^2}{8}y_0^5 - \frac{\epsilon}{4}y_0 v_0^2. \tag{76}$$

All together, these results provide us with an expansion of $y(t)$ out to fourth order in time. As a specific example, let's consider the initial condition

$$v_0 = 0, \tag{77}$$

along with some arbitrary initial position y_0 - that is, we let the spring go from rest. In this case, we find

$$y_1 = 0\ ;\ y_2 = -\frac{y_0}{2}\left(\omega^2 + \epsilon y_0^2\right)\ ;\ y_3 = 0\ ;\ y_4 = \frac{y_0}{24}\left(\omega^4 + 4\epsilon\omega^2 y_0^2 + 3\epsilon^2 y_0^4\right), \tag{78}$$

which gives

$$y(t) \approx y_0 - \frac{y_0}{2}\left(\omega^2 + \epsilon y_0^2\right)t^2 + \frac{y_0}{24}\left(\omega^4 + 4\epsilon\omega^2 y_0^2 + 3\epsilon^2 y_0^4\right)t^4 \tag{79}$$

For small enough times, this solution should be a reasonably good approximation to the full motion of the spring.

However, the obvious disadvantage to this approach is that it does not tell us anything about very long times - we know that whenever we keep a Taylor series expansion to a finite number of terms, eventually, after long enough time, the series expansion should become a worse and worse approximation to the true functional form. In fact, for very large times, our solution behaves as

$$y\left(t \to \infty\right) \to \pm\infty, \tag{80}$$

with the sign depending on the sign of y_0. This behaviour is quite general, since no matter what order we take our expansion to,

$$y\left(t\right) = \sum_{n=0}^{N} y_n t^n, \tag{81}$$

so long as N is finite, we will eventually have

$$y\left(t\right) \approx y_N t^N \tag{82}$$

for large enough times, which again diverges to infinity. Again, we know this is not consistent with the physics of the problem - the particle should oscillate back and forth forever, with a finite oscillation amplitude.

However, once again, we can use our prior knowledge about the periodic motion of the particle to help us improve our perturbative approach. We know that the period of oscillation of the particle must be

$$T = \sqrt{2m}\int_{y_-}^{y_+} \frac{dx}{\sqrt{E - U\left(x\right)}}. \tag{83}$$

While we may not be able to do this integral in closed form, for any set of initial conditions, it is certainly easy to calculate a numerical value for the integral using a calculator. Once we have computed T, we can then use it to impose a constraint on the particle's motion,

$$y\left(T\right) = y\left(0\right). \tag{84}$$

All further motion of the particle after time T simply repeats exactly the same basic oscillation, such that

$$y\left(t + nT\right) = y\left(t\right). \tag{85}$$

For this reason, our series expansion of $y\left(t\right)$ only needs to be accurate up until the time T in order to understand the full motion of the particle. Furthermore, we can also get a good sense for how accurate our perturbative result is, by checking to see how well it satisfies the constraint

$$y\left(T\right) \approx y\left(0\right). \tag{86}$$

If our perturbative result satisfies this constraint very well, we know that we have taken enough terms in the expansion to get a reasonably good approximation to the particle's motion. If the constraint is satisfied very poorly, we know we need to take more terms in our expansion.

Figure 6 shows an example solution using this approach, compared with the result of a more sophisticated numerical algorithm. Notice that for short enough times, the two curves agree well, but as time goes on, the two curves start to deviate noticeably. Figures 7 and 8 show the two curves over longer periods of time, where it becomes obvious that our fourth-order approximation becomes relatively poor before even a single period of oscillation has occurred. Unfortunately, we usually need to take quite a few terms in order for this type of perturbation theory technique to be accurate, and in many situations it is more practical to use a numerical algorithm to perform the time evolution. However, the one advantage of this approach is that to an arbitrarily high level of precision, we can find a closed-form expression for our particle's motion, in which the dependence on the initial conditions is explicit. Performing the algebraic manipulations required in working out the Taylor series coefficients is something that a program like Mathematica can help us with.

For systems which exhibit periodic behaviour, the short time approximation method can be used to help us understand the motion of our system, even when the nonlinear terms in our system are not small. However, for systems which are not periodic and do not have small nonlinearities, we are generally in a much trickier situation if we want to know something about the long-time behaviour of the system. While there are some techniques available for dealing with situations like this, they are unfortunately beyond the scope of our class.

Chaos

In the absence of damping and external driving forces, we have seen that our system exhibits regular periodic motion, and we have derived two different techniques for understanding the nature of this periodic motion. In general, though,

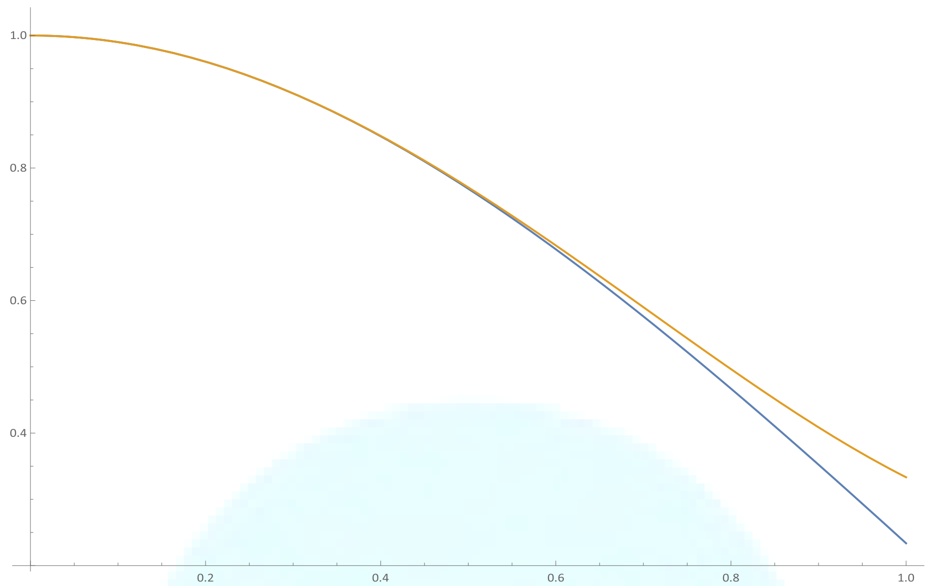

Figure 6: A plot of the motion of our particle, using the short time approximation, for $k = m = \epsilon = y_0 = 1$ (orange curve). The blue curve shows the result of a more sophisticated numerical calculation. Notice that for short times the two curves agree well, although at longer time, the deviation starts to become noticeable.

we may want to know something about the solutions to the full nonlinear differential equation,

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y + \phi y^2 + \epsilon y^3 = f(t), \tag{87}$$

What type of behaviour does this equation exhibit, for an arbitrary forcing function?

The answer to this question is that there is, in fact, an enormous range of different types of qualitative behaviour which can arise from this differential equation. Linear systems are special in the sense that the superposition rule allows us to more or less understand all of the possible solutions to a linear differential equation, for any arbitrary set of initial conditions. This special property is generically not true, however, for nonlinear systems, and any attempt to understand the entire range of behaviour for an arbitrary set of initial conditions is doomed to failure. For this reason, it is impossible to give a short summary here of all of the interesting types of behaviour this equation admits.

However, one of the most striking features of this equation, which is common to almost all nonlinear systems, is that it is capable of exhibiting **chaos**. Chaotic motion occurs when the trajectory of a system is highly sensitive to its initial conditions. To demonstrate what I mean by this, consider two solutions to my differential equation, x_1 and x_2, with slightly different initial conditions. The

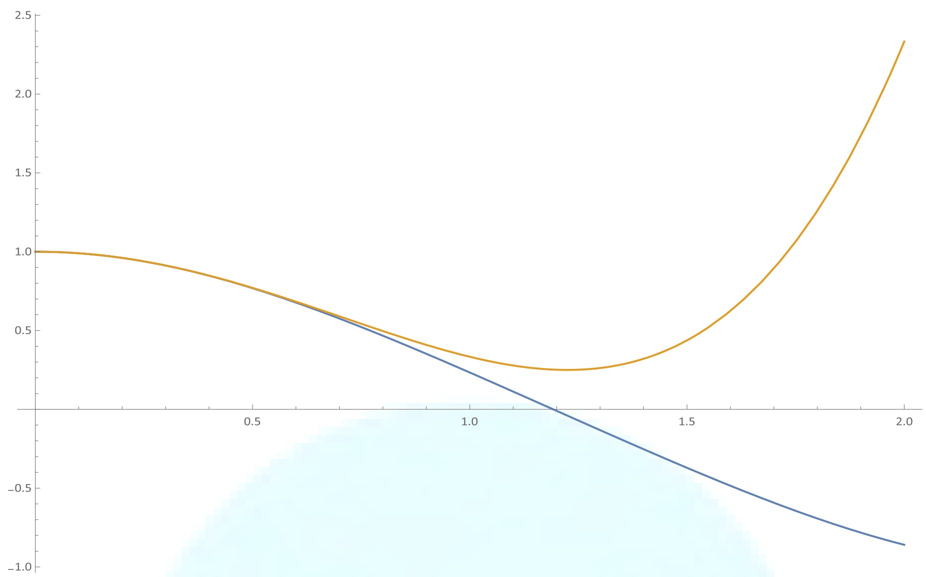

Figure 7: A plot of the motion of our particle, using the short time approximation, for $k = m = \epsilon = y_0 = 1$ (orange curve). The blue curve shows the result of a more sophisticated numerical calculation. Notice that for short times the two curves agree well, although at longer time, the deviation starts to become noticeable.

difference between these two functions, as a function of time, is given by

$$\Delta x(t) = x_1(t) - x_2(t). \tag{88}$$

The question I may now ask is, if the difference between these two solutions is small at time zero,

$$\Delta x(0) \ll 1\ ,\ \ \Delta x'(0) \ll 1, \tag{89}$$

then how does $\Delta x(t)$ behave at large times?

For linear systems, the difference between two solutions is typically either a constant, or decays to zero. For example, in the linear harmonic oscillator, the difference between two solutions with zero initial velocity, and slightly different starting positions, is given by

$$\Delta x(t) = (A_1 - A_2)\cos(\omega t). \tag{90}$$

The closer the initial conditions, the smaller this quantity. For this reason, we say that the differential equation describing the linear oscillator demonstrates **stability -** two solutions which are are initially close to each other will **stay** close to each other, for all time.

However, in stark contrast to this is chaotic behaviour. In a chaotic system, the difference between two solutions, even those which are initially separated

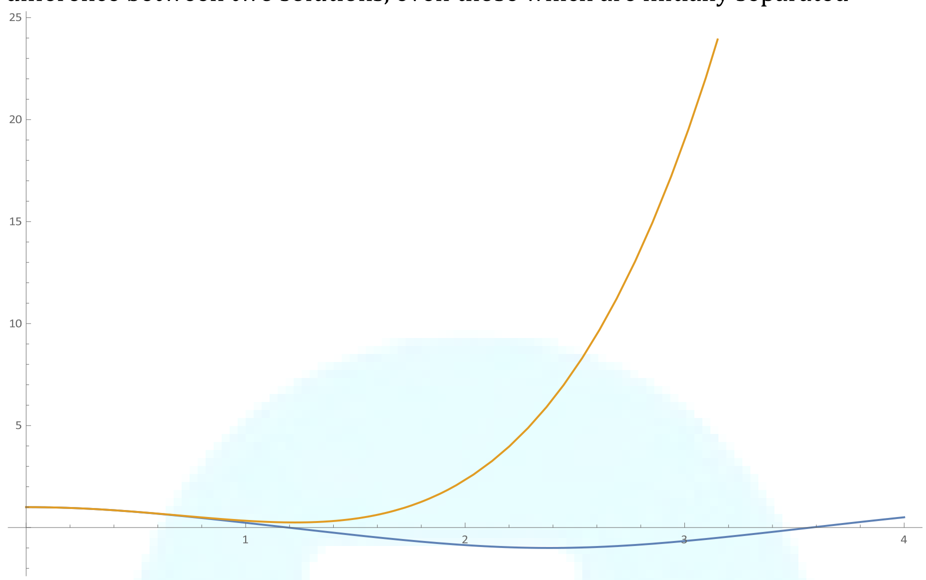

Figure 8: A plot of the motion of our particle, using the short time approximation, for $k = m = \epsilon = y_0 = 1$ (orange curve). The blue curve shows the result of a more sophisticated numerical calculation. Notice that for short times the two curves agree well, although at longer time, the deviation starts to become noticeable.

by a very small amount, can diverge exponentially,

$$\Delta x(t) \sim \Delta x_0 e^{\lambda t}. \tag{91}$$

The quantity λ is known as the **Lyapunov exponent**, and quantifies the extent to which a system is chaotic. Almost all nonlinear systems are capable of demonstrating chaotic behaviour, although this behaviour may not occur across the entire phase space - for example, in the forced nonlinear oscillator, only certain forcing functions will result in chaotic behaviour. While a driving force is necessary to elicit chaotic behaviour in the nonlinear oscillator, other nonlinear systems (some of which involve conservative forces and thus conservation of energy) can demonstrate chaotic behaviour even without an external driving force.

Whether or not a system is capable of demonstrating chaos is incredibly important, for a variety of reasons. One of the most important reasons is that a lack of stability in a system means that modelling its long time behaviour can be very difficult. The reason for this is that any realistic system which we study in a laboratory will have some initial conditions that we can measure only to within some experimental precision. For this reason, we will not know precisely what the initial conditions of our system are, and so if the system demonstrates chaos, it will be essentially impossible to say anything about its long-time behaviour, since our predicted motion can diverge exponentially from the true motion. This is why systems which are, in principle, deterministic, can still be, for all practical purposes, impossible to simulate. It's also, more or less, the reason why I can't tell you what the weather will be like one month from now.

Secondly, chaos is important because chaotic systems are capable of a behaviour known as **thermalization**. I know that if I take a complicated system of particles and leave them in a box, eventually, after enough time, the gas inside of the box will obey the laws of thermodynamics, regardless of the initial conditions for each and every one of the particles in the box. The assumption that this will occur is known as the **ergodic hypothesis**, and it is the fundamental assumption behind all of statistical mechanics. For this reason, it is important to understand exactly when a many-body system is capable of demonstrating chaos. While this question was settled for classical systems many years ago, it is in fact still an active research question as to the precise conditions necessary for a closed, **quantum** system to come to thermal equilibrium, and in fact, this question is at the heart of my own research.

While nonlinear systems are very interesting, they also tend to be very challenging to solve, and for this reason, we will not be able to dedicate any more time to their study in this class. However, for those of you interested in learning more about nonlinear systems, the course Physics 106 here at UCSB is dedicated entirely to this subject.