Orbital angular momentum

Consider a particle of mass m, and momentum $\vec{p}$ and position vector $\vec{r}$. In classical mechanics, the particle's orbital angular momentum is given by a vector $\vec{L}$, defined by

$$\vec{L} = \vec{r} \times \vec{p}$$

This vector points in a direction that is perpendicular to the plain containing $\vec{r}$ and $\vec{p}$, and has a magnitude L = rp sinα , where α is the angle between $\vec{r}$ and $\vec{p}$. In Cartesian coordinates, the components of $\vec{L}$ are

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

The corresponding QM operators representing L_x, L_y, L_z are obtained by replacing x, y, z and p_x, p_y, p_z with corresponding QM operators, giving

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

In more compact form, this can be written as a Vector operator

$$\vec{L} = -i\hbar\,(\vec{r} \times \vec{\nabla})$$

It is easy to prove that $\vec{L}$ is Hermitian.

Angular momentum Algebra

Using the commutation relations derived for $\vec{r}$ *and* $\vec{p}$, the commutation relation for different components of $\vec{L}$ are readily derived.

$$[L_x, L_y] = [(yp_z - zp_y), zp_x - xp_z]$$

$$= [yp_z, zp_x] + [zp_y, xp_z] + [yp_z, xp_z] + [zp_y, zp_x]$$

Since y and p_z commute with each other and with z and p_z, the first term becomes

$$[yp_z, zp_x] = yp_z zp_x - zp_x yp_z = yp_x [p_z, z] = -i\hbar\, yp_z$$

Similarly the second commutator becomes

$$[zp_y, xp_z] = zp_y xp_z - xp_z zp_y = xp_y [z, p_z] = i\hbar xp_y$$

The third ans forth commutator vanishes , thus we find that

$[L_x, L_y] = i\hbar\,[xp_y - yp_x] = i\hbar L_x$

Similarly we can show that

$[L_y, L_z] = i\hbar L_z$
$[L_z, L_x] = i\hbar L_y$

The three equations are similar to the relation

$$\vec{L} \times \vec{L} = i\hbar\vec{L}$$

The fact that the operators representing the different components of the angular momentum do not commute, implies it is impossible to obtain definite values for all component of the angular momentum when measured simultaneously. This means that if the system is in eigenstate of one component of angular momentum, it will in general will not be an eigenstate of either of the other two components.

Commutation Relations

1. $[\hat{x}, \widehat{L_x}] = [\hat{x}, yp_z - zp_y]$

$= [\hat{x}, yp_z] - [\hat{x}, zp_y]$

$= 0$

2. $[\hat{y}, L_y] = 0$

3. $[\hat{z}, L_z] = 0$

4. $[y, L_x] = [y, yp_z - zp_y]$

$= [y, yp_z] - [y, zp_y]$

$= -z[y, p_z]$

$= -i\hbar z$

5. $[L_x, y] = i\hbar z$

6. $[L_y, z] = i\hbar x$

7. $[L_z, x] = i\hbar y$

8. $[p_x, L_x] = [p_x, yp_z - zp_y]$

$= 0$

9. $[p_y, L_y] = 0$

10. $[p_z, L_z] = 0$

11. $[p_x, L_y] = [p_x, zp_x - xp_z]$

$= -[p_x, x]\, p_z$

$= i\hbar p_z$

12. $[p_y, L_z] = i\hbar p_x$

13. $[p_z, L_x] = i\hbar p_y$

14. $[L_x, L_y L_z] = [L_x, L_y] L_z + L_y [L_x, L_z]$

$= i\hbar L_z L_z + L_y i\hbar L_y$

$= i\hbar [L_z^2 + L_y^2]$

15. $[L_x^2, y] = [L_x L_x, y]$

$= [L_x, y] L_x + L_x [L_x, y]$

$= i\hbar z L_x + i\hbar L_x z$

$= i\hbar [z L_x + L_x z]$

Raising operator and lowering operator

The operator L_+ is called raising operator and L_- is known as lowering operator.

$L_+ = L_x + iL_y$

Also $L_+ |l, m\rangle = C_+ |l, m+1\rangle$

$L_- = L_x - iL_y$

$L_- |l, m\rangle = C_- |l, m-1\rangle$

$(L_+)^+ = L_-$

$(L_-)^+ = L_+$

Commutation relation

1. $[L_+, L_x] = [L_x + iL_y, L_x]$

$= [L_x, L_x] + i[L_y, L_x]$

$= 0 + i \times -i\hbar L_z$

$= \hbar L_z$

2. $[L_-, L_y] = i\hbar L_z$

3. $[L_+, L_-] = 2\hbar L_z$

4. $[L_y, L_+] = -i\hbar L_z$

5. $L_+ L_- = L^2 - L_z^2 + \hbar L_z$

6. $L_- L_+ = L^2 - L_z^2 - \hbar L_z$