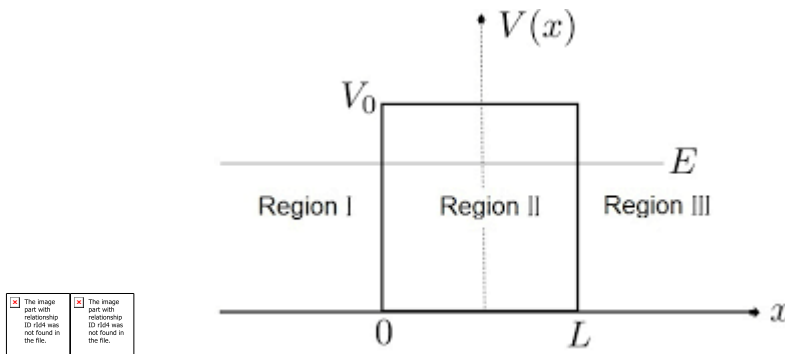


Quantum mechanical tunneling



The potential $V(x)$ is defined by

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 < x < L \\ 0 & x > 0 \end{cases}$$

Consider a stream of particles of mass m approaching the square barrier from the left. Classically the energy of the particle $E < V_0$ it is always reflected where as it is transmitted if $E > V_0$. However, quantum mechanically it can be seen that there is always a finite probability for a particle to penetrate or leak through the barrier and continue its forward motion even if $E < V_0$. This phenomenon is called quantum mechanical tunneling, is possible because of the wave nature of matter.

Case (I) $E < V_0$

The Schrödinger equation for region (1) is given by

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{2mE}{\hbar^2} \psi_1 = 0 \rightarrow (1)$$

$$\text{Put } k^2 = \frac{2mE}{\hbar^2}$$

$$(1) \rightarrow \frac{\partial^2 \psi_1}{\partial x^2} = -K^2 \psi_1$$

$$\text{On solving this } (D^2 + k^2)\psi_1 = 0$$

$$D^2 = -K^2$$

$$D = \pm ik$$

$$\text{Therefore } \psi_1 = Ae^{ikx} + Be^{-ikx} \rightarrow (2)$$

The first term represents the incident wave and second term represents the reflected wave.

$$\Psi_i = Ae^{ikx}$$

$$\Psi_r = Be^{-ikx}$$

In region 2

$$\frac{\partial^2 \Psi_2}{\partial x^2} + \frac{2m(V_0 - E)}{\hbar^2} \Psi \quad \text{here } \alpha^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

On solving this we get

$$\Psi_2 = B e^{\alpha x} + C e^{-\alpha x}$$

For $x \geq L$

$$\frac{\partial^2 \Psi_3}{\partial x^2} = -k^2 \Psi_3 \rightarrow (3)$$

The probability density of the incident, reflected and transmitted waves are 1, $|A|^2$ and $|D|^2$, respectively. Consequently the transmission coefficient $T = |D|^2$ and the reflection coefficient $R = |A|^2$. The two are connected by the relation $R+T = 1$. The continuity conditions on the wave functions and their first derivative at $x = 0$ and $x = l$ give

$$1+A = B+C \rightarrow (4)$$

$$ik - ikA = \alpha B - \alpha C \rightarrow (5)$$

And

$$B e^{\alpha l} + C e^{-\alpha l} = D e^{ikl} \rightarrow (7)$$

$$\alpha B e^{\alpha l} - \alpha C e^{-\alpha l} = ik D e^{ikl} \rightarrow (8)$$

On solving these equations we get

$$B = \frac{D}{2\alpha} (\alpha + ik) e^{ikl - \alpha l} \rightarrow (9)$$

$$C = \frac{D}{2\alpha} (\alpha - ik) e^{ikl + \alpha l} \rightarrow (10)$$

On solving (4) and (9) we get,

$$A = \frac{D(\alpha + ik)}{(\alpha - ik)} e^{ikl - \alpha l} - \frac{\alpha + ik}{\alpha - ik}$$

Substituting A, B, C in equ (4) gives

$$D = \frac{2ik\alpha e^{-ikl}}{(\alpha^2 - k^2)^2 \sinh(\alpha l) - 2iak \cosh(\alpha l)}$$

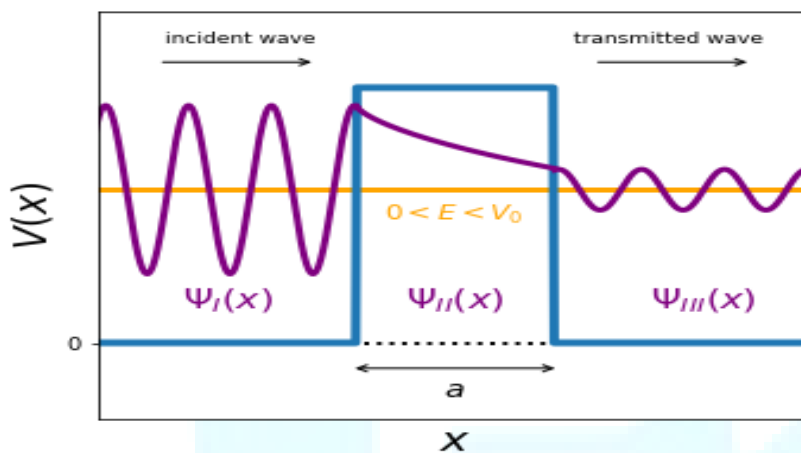
Now $T = |D|^2$ and also assuming that for broad high barrier

$$\alpha a \gg 1 \text{ and } \sinh(\alpha l) = \cosh(\alpha l) \rightarrow \frac{1}{2} e^{l\alpha}$$

So
$$T = \frac{16k^2 \alpha^2 e^{-2l\alpha}}{(\alpha^2 + k^2)^2}$$

Substituting the values of α^2 and k^2 gives

$$T = \frac{16E(V_0 - E)e^{-2\alpha l}}{V_0^2}$$



An illustration of the wave function in three regions