Rotations and the Euler angles

1 Rotations

Consider two right-handed systems of coordinates, XYZ and $x_1x_2x_3$, rotated arbitrarily with respect to one another (see Fig. →). We would like to be able to link easily the coordinates of any vector $\vec{A}$ in the two frames of reference. Let $\vec{e}_X, \vec{e}_Y, \vec{e}_Z$ be the unit vectors for the axes of the first system, and $\vec{e}_1, \vec{e}_2, \vec{e}_3$ the unit vectors for the axes of the second system. Then, by definition:

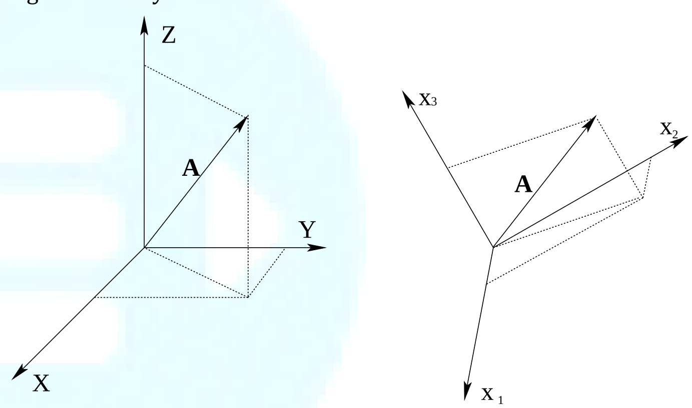
$$\vec{A} = A_X\vec{e}_X + A_Y\vec{e}_Y + A_Z\vec{e}_Z$$

Then, we can express one set of projections in terms of the other one:

and

$$\vec{A} = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$$

Fig 1. Projection of the same vector $\vec{A}$ onto two different righthanded systems of coordinates.



$$A_1 = \vec{e}_1 \cdot \vec{A} = (\vec{e}_1 \cdot \vec{e}_X)A_X + (\vec{e}_1 \cdot \vec{e}_Y)A_Y + (\vec{e}_1 \cdot \vec{e}_Z)A_Z$$

$$A_2 = \vec{e}_1 \cdot \vec{A} = (\vec{e}_2 \cdot \vec{e}_X)A_X + (\vec{e}_2 \cdot \vec{e}_Y)A_Y + (\vec{e}_2 \cdot \vec{e}_Z)A_Z$$

$$A_3 = \vec{e}_1 \cdot \vec{A} = (\vec{e}_3 \cdot \vec{e}_X)A_X + (\vec{e}_3 \cdot \vec{e}_Y)A_Y + (\vec{e}_3 \cdot \vec{e}_Z)A_Z$$

or, in matrix form:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_X & \vec{e}_1 \cdot \vec{e}_Y & \vec{e}_1 \cdot \vec{e}_Z \\ \vec{e}_2 \cdot \vec{e}_X & \vec{e}_2 \cdot \vec{e}_Y & \vec{e}_2 \cdot \vec{e}_Z \\ \vec{e}_3 \cdot \vec{e}_X & \vec{e}_3 \cdot \vec{e}_Y & \vec{e}_3 \cdot \vec{e}_Z \end{pmatrix} \cdot \begin{pmatrix} A_X \\ A_Y \\ A_Z \end{pmatrix} \quad (1)$$

Let us analyze the elements of the 3 × 3 matrix. By definition $\vec{e}_1 \cdot \vec{e}_X = \cos\varphi_{1X}$, where φ_{1X} is the angle between the two unit vectors $\vec{e}_1$ and $\vec{e}_X$. Similarly, all other elements of this matrix depend only on the various angles between various sets of axes, but are independent of the projected vector $\vec{A}$. It follows that for any other vector $\vec{B}$, we will have automatically:

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_X & \vec{e}_1 \cdot \vec{e}_Y & \vec{e}_1 \cdot \vec{e}_Z \\ \vec{e}_2 \cdot \vec{e}_X & \vec{e}_2 \cdot \vec{e}_Y & \vec{e}_2 \cdot \vec{e}_Z \\ \vec{e}_3 \cdot \vec{e}_X & \vec{e}_3 \cdot \vec{e}_Y & \vec{e}_3 \cdot \vec{e}_Z \end{pmatrix} \cdot \begin{pmatrix} B_X \\ B_Y \\ B_Z \end{pmatrix}$$

In other words, if we know the 3 × 3 matrix, then we can find the components of any vector in one of the systems, if we know them in the other.

Before continuing, let us introduce some simpler notation. We will denote

$$\vec{A}_{XYZ} = \hat{R}^{-1} \cdot \vec{A}_{body}$$

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \vec{A}_{body}, \qquad \begin{pmatrix} A_X \\ A_Y \\ A_Z \end{pmatrix} = \vec{A}_{XYZ} \quad \text{and} \quad \hat{R} = \begin{pmatrix} \vec{e}_1\cdot\vec{e}_X & \vec{e}_1\cdot\vec{e}_Y & \vec{e}_1\cdot\vec{e}_Z \\ \vec{e}_2\cdot\vec{e}_X & \vec{e}_2\cdot\vec{e}_Y & \vec{e}_2\cdot\vec{e}_Z \\ \vec{e}_3\cdot\vec{e}_X & \vec{e}_3\cdot\vec{e}_Y & \vec{e}_3\cdot\vec{e}_Z \end{pmatrix}$$

and therefore we have $\vec{A}_{body} = \hat{R}\cdot\vec{A}_{XYZ}$. It then follows that:
where $R\hat{}^{-1}$ is the inverse of matrix $R\hat{}$, and it should be clear that its matrix elements are:

$$\hat{R}^{-1} = \begin{pmatrix} \vec{e}_X\cdot\vec{e}_1 & \vec{e}_X\cdot\vec{e}_2 & \vec{e}_X\cdot\vec{e}_3 \\ \vec{e}_Y\cdot\vec{e}_1 & \vec{e}_Y\cdot\vec{e}_2 & \vec{e}_Y\cdot\vec{e}_3 \\ \vec{e}_Z\cdot\vec{e}_1 & \vec{e}_Z\cdot\vec{e}_3 & \vec{e}_Z\cdot\vec{e}_3 \end{pmatrix}$$

If it's not clear, then derive them and check!

We can see that the matrix $R\hat{}^{-1}$ is just the transpose of matrix $R\hat{}$ (by definition, M is the transpose of N, i.e. $M = N^T$, if $m_{ij} = n_{ji}$ for all i,j). This property is a consequence of the invariance of the length of any vector under rotations. If we denote:

$$\vec{A}^T_{body} = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}; \qquad \vec{A}^T_{XYZ} = \begin{pmatrix} A_X & A_Y & A_Z \end{pmatrix}$$

then

$$|\vec{A}|^2 = \vec{A}\cdot\vec{A} = A_XA_X + A_YA_Y + A_ZA_Z = \vec{A}^T_{XYZ}\cdot\vec{A}_{XYZ} = \vec{A}^T_{body}\cdot\vec{A}_{body}$$

(the vector has the same length in any system of coordinates). But $\vec{A}_{body} = \hat{R}\cdot\vec{A}_{XYZ} \rightarrow \vec{A}^T_{body} =$ $A^{\sim T}_{XYZ} \cdot R^{\hat{}T}$ (this last property can be checked easily using the definition of the transposed matrix), and therefore:

$$\vec{A}^T_{body}\cdot\vec{A}_{body} = \left(\vec{A}^T_{XYZ}\cdot\hat{R}^T\right)\left(\hat{R}\cdot\vec{A}_{XYZ}\right) = \vec{A}^T_{XYZ}\left(\hat{R}^T\cdot\hat{R}\right)\vec{A}_{XYZ}$$

which implies that $R^{\hat{}T}\cdot R\hat{} = 1 \rightarrow R^{\hat{}T} = R\hat{}^{-1}$.

This property is extremely useful, since it allows us to easily find the inverse of any rotation matrix, by just taking its transpose.

2 Rotation about one axis

Let us derive the expression of $R\hat{}_3$ for the case where the axes $0Z$ and $0x_3$ are parallel, and the sets of axes XY and x_1x_2 are rotated by an angle φ with respect to one another (see Fig). In this case, we know (see, for instance, discussion of polar coordinates) that the relationship between the unit vectors is:

$$\begin{cases} \vec{e}_1 = \cos\phi\vec{e}_X + \sin\phi\vec{e}_Y \\ \vec{e}_2 = -\sin\phi\vec{e}_X + \cos\phi eY \\ \vec{e}_3 = \vec{e}_Z \end{cases}$$

We can now compute the various dot products; for instance $\sim e_1 \cdot\sim e_X = \cos\varphi$, etc, and we find

$$\hat{R}_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

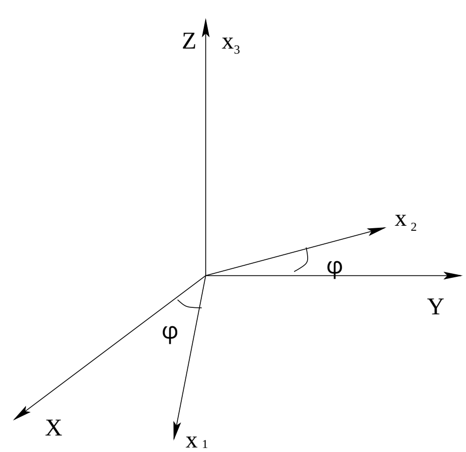


Fig 2. Rotation by an angle φ about the axis $Oz = Ox_3$.

You should check that $\hat{R}_3(\varphi_1)\hat{R}_3(\varphi_2) = \hat{R}_3(\varphi_1 + \varphi_2)$ – meaning that if I rotate first by angle φ_2 followed by a rotation by angle φ_1 (about the same axis!) it's as if I did a single rotation by angle $\varphi_1 + \varphi_2$. Which is true.

The inverse matrix is then:

$$\hat{R}_3^{-1}(\phi) = \hat{R}_3^T(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hat{R}_3(-\phi)$$

This makes perfect sense as well; if system 123 is rotated with $+\varphi$ with respect to system XYZ, then system XYZ is rotated with $-\varphi$ with respect to 123. As a result, the rotation matrices should have the same form with $\varphi \to -\varphi$, and that is precisely what we found.

In the same way, we can write down the matrices for rotations about any other axis. For instance, if OX and $0x_1$ are kept parallel and we perform a rotation by an angle θ about them, we find

$$\hat{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

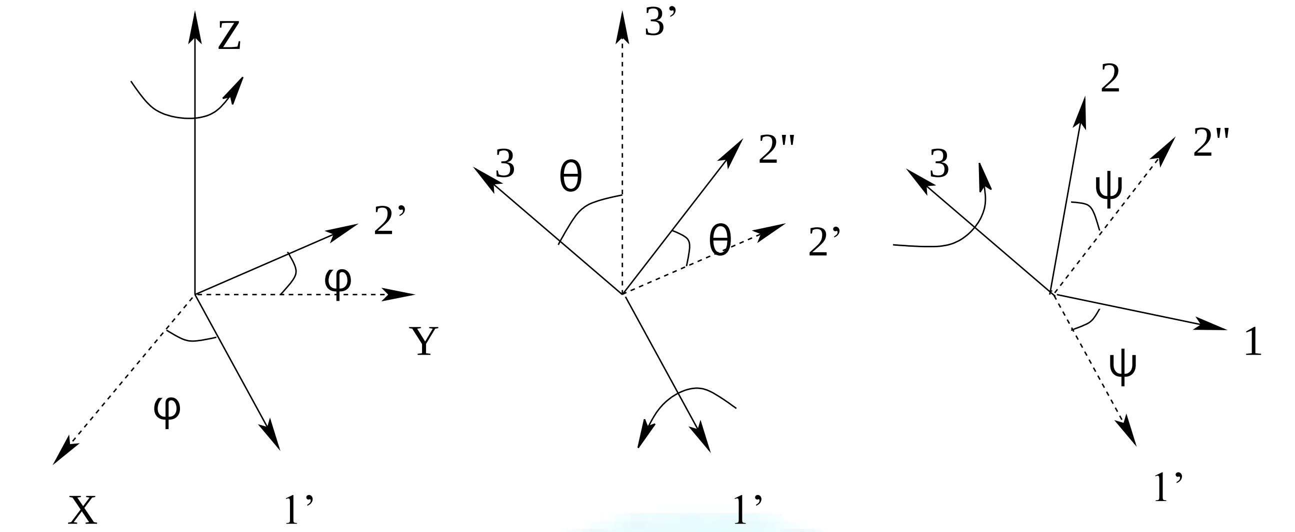
We can now use the fact that any general 3D rotation can be decomposed into a product of 3 rotations about 3 different axes, to find the form of a general rotation matrix.

3 Euler's angles

We characterize a general orientation of the "body" system $x_1x_2x_3$ with respect to the inertial system XYZ in terms of the following 3 rotations:

1. rotation by angle φ about the Zaxis;
2. rotation by angle θ about the new x'_1 axis, which we will call the line of nodes ;
3. rotation by angle ψ about the new x_3 axis.

These rotations are illustrated in the following figure:



We can now write the general rotation matrix that links $\vec{A}_{body}$ with $\vec{A}_{XYZ}$ as the product of the 3 rotations about the corresponding axes:

$$\hat{R}(\phi,\theta,\psi) = \hat{R}_3(\psi)\cdot\hat{R}_1(\theta)\cdot\hat{R}_3(\phi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\cdot\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

leading to the rather ugly general formula:

$$\hat{R}(\phi,\theta,\psi) = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

Fortunately, we will never need to use this matrix. All we really need is to be able to write the components of the angular velocity Ω in both systems of coordinates. Since˜ Ω describes precisely˜ how fast the angles vary in time, we have:

$$\vec{\Omega} = \frac{d\vec{\phi}}{dt} + \frac{d\vec{\theta}}{dt} + \frac{d\vec{\Psi}}{dt} = \dot{\phi}\vec{e}_Z + \dot{\theta}\vec{e}_{1'} + \dot{\psi}\vec{e}_3$$

since the three rotations are about these particular axes. Let us analyze each contribution to Ω.˜

1. $\dot{\vec{\varphi}} = \vec{e}_Z\dot{\varphi}$ (with respect to XYZ system). Following the rotations, we find that with respect to 123 system, we have:

$$\vec{e}_Z = \cos\theta\vec{e}_3 + \sin\theta\vec{e}_{2''} = \cos\theta\vec{e}_3 + \sin\theta(\sin\psi\vec{e}_1 + \cos\psi\vec{e}_2)$$

and therefore: $\dot{\vec{\varphi}} = \sin\theta\sin\psi\dot{\varphi}\vec{e}_1 + \sin\theta\cos\psi\dot{\varphi}\vec{e}_2 + \cos\theta\dot{\varphi}\vec{e}_3$

2. $\dot{\vec{\theta}} = \vec{e}_{1'}\dot{\theta} = \dot{\theta}(\cos\varphi\vec{e}_X + \sin\varphi\vec{e}_Y)$ (with respect to XYZ), whereas

$$\vec{e}_{1'} = \cos\psi\vec{e}_1 - \sin\psi\vec{e}_2 \rightarrow \dot{\vec{\theta}} = \cos\psi\dot{\theta}\vec{e}_1 - \sin\psi\dot{\theta}\vec{e}_2$$

with respect to 123.

3. $\dot{\vec{\psi}} = \vec{e}_3\dot{\psi}$ (with respect to 123), whereas

$$\vec{e}_3 = \cos\theta\vec{e}_{3'} - \sin\theta\vec{e}_{2'} = \cos\theta\vec{e}_Z - \sin\theta(-\sin\varphi\vec{e}_X + \cos\varphi\vec{e}_Y) \rightarrow$$

$$\dot{\vec{\psi}} = \sin\theta\sin\varphi\dot{\psi}\vec{e}_X - \sin\theta\cos\varphi\dot{\psi}\vec{e}_Y + \cos\theta\dot{\psi}\vec{e}_Z$$

Adding all three components together, we find that, with respect to the body reference system,

$$\vec{\Omega} = (\sin\theta\sin\psi\dot{\phi} + \cos\psi\dot{\theta})\vec{e}_1 + (\sin\theta\cos\psi\dot{\phi} - \sin\psi\dot{\theta})\vec{e}_2 + (\cos\theta\dot{\phi} + \dot{\psi})\vec{e}_3 \quad (2)$$

while with respect to the inertial reference system:

$$\vec{\Omega} = (\cos\phi\dot{\theta} + \sin\theta\sin\phi\dot{\psi})\vec{e}_X + (\sin\phi\dot{\theta} - \sin\theta\cos\phi\dot{\psi})\vec{e}_Y + (\dot{\phi} + \cos\theta\dot{\psi})\vec{e}_Z$$

So if we can solve the EL equations and find how these angles vary in time, we can figure out what's the angular speed in either of the two reference systems.

4 Kinetic energy in terms of Euler's angles

Let us choose the CM as the reference point O, and we will choose the principal axes of inertia as the body reference frame. The total kinetic energy of the object will be:

$$\mathcal{T} = \frac{1}{2}M\vec{V}_{CM}^2 + \frac{1}{2}\vec{\Omega}\cdot\hat{I}_{CM}\cdot\vec{\Omega} = \frac{1}{2}M\vec{V}_{CM}^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^3 + I_3\Omega_3^2\right)$$

where $\Omega_1 = \sin\theta\sin\psi\dot{\varphi} + \cos\psi\dot{\theta}$, etc [see Eq. (2)].

For an asymmetric top, the general formula is rather complicated, and we will not use it. For a symmetric top with $I_1 = I_2 \neq I_3$, if you put the expressions for Ω_1,Ω_2 and Ω_3 in and simplify a bit, you find:

$$\mathcal{L} = \frac{1}{2}M\vec{V}_{CM}^2 + \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - U(\vec{R}_{CM},\phi,\theta,\psi)$$

This is our Lagrangian in terms of our 6 generalized coordinates, namely $\vec{R}_{CM},\theta,\phi,\psi$.