ENTRI

SETS AND FUNCTIONS

CARDINALITY

Let A be a finite set, the number of elements in set A is known as cardinality of A, usually denoted as $n(A)$.

RESULT

Let A, B, C be finite sets, then
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
- $n(A/B) = n(A) - n(A \cap B)$
- $n(A\Delta B) = n(A) + n(B) - 2n(A \cap B) = n(A \cup B) - n(A \cap B)$
- $n(A \times B) = n(A).\, n(B)$
- $n(A) = m \Rightarrow n(P(A)) = 2m$, where $P(A)$ is the power set of A.

RELATION

A relation R from a set A to B is a subset of $A \times B$, ie, $R \subseteq A \times B$. If $(a, b) \in R$, then we say that "a is related to b (aRb)" or a~b.

• If $n(A) = m$, $n(B) = n$, then the number of relations from $A \rightarrow B$ is 2^{mn}

• Number of relations on A of cardinality n is 2^{n^2}

NOTE

- $R: A \rightarrow A$ is said to be Reflexive relation if $(a, a) \in R$, $\forall a \in A$.
  Number of reflexive relations on A of cardinality m is 2^{m^2-m}
- $R: A \rightarrow A$ is said to be Symmetric relation if $(a, b) \in R \Rightarrow (b, a) \in R$
  Reflexive relation need not be symmetric and vice versa.
  Number of symmetric relations on A of cardinality m is $2^{\frac{m(m+1)}{2}}$
- $R: A \rightarrow A$ is said to be Anti-symmetric relation on a set A if $(a, b) \in R$ & $(b, a) \in R \Leftrightarrow a = b$.
- $R: A \rightarrow A$ is said to be a Transitive relation on a set A. if $(a, b) \in R$ & $(b, c) \in R \Rightarrow (a, c) \in R$.

EQUIVALENCE RELATION

If the relation R is reflexive, symmetric, transitive, then R is said to be an equivalence relation.

FUNCTIONS

Let $A, B \subset \mathbb{R}$ and let $f: A \rightarrow B$ be a relation from A to B, then f is said to be a function from A to B if each element of A is related to a unique element in B.

Here A is said to be the Domain D(f) of the function f and B is the Codomain of f.
If $b \in B$ is the unique element related from $a \in A$ then we say that b is the Image of a and write $f(a) = b$, In this case a is known as a preimage of b. The set of all such $b = f(a)$, $a \in A$ is called the Range of f denoted by $R(f)$. f is said to be real valued if $R(f) \subseteq \mathbb{R}$

EXAMPLE

- $f(x) = \sqrt{x}$, $D(f) = \mathbb{R}_+ \cup \{0\} = [0, \infty) = R(f)$
- $f(x) = k$, $D(f) = \mathbb{R}$, $R(f) = \{k\}$
- $f(x) = x$, $D(f) = \mathbb{R}$, $R(f) = \mathbb{R}$
- $f(x) = a_0 + a_1x + a_2x_2 + \cdots + a_nx_n$, $a_i \in \mathbb{R}$, $D(f) = \mathbb{R}$

CLASSIFICATION OF FUNCTIONS

- function f is said to be a one-one function if, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $x$for any $x_1, x_2 \in D(f)$. Failure of the converse leads the function to be Many one.
- A function $f: A \to B$ is said to be a onto function if, $f(A) = B$. Otherwise, function is just from A into B.
- A function which is both one-one and onto is known as a Bijection

COMPOSITION OF FUNCTIONS

Let f, g be function such that $f: A \to B$, $g: B \to C$, then their composition $g \circ f$: A → C can be defined by $g \circ f(x) = g(f(x))$, $\forall x \in A$.

NOTE

- In general $g \circ f \neq f \circ g$, (later one is defined only if $B = C$).
- $g \circ f, f \circ g, f \circ f, g \circ g$ all are defined together only if $A = B = C$

INVERSE OF A FUNCTION

A function $f: A \to B$ is to be invertible if there is a function $g: B \to A$ so that $f \circ g = I_B$ & $g \circ f = I_A$, where I_A is the identity function on the set A.

RESULTS

- f is Invertible $\Leftrightarrow$ f is a Bijection.
- Let A, B be sets with cardinalities m and n are respectively, then ∃ a bijection $f: A \to B \Leftrightarrow n = m$.
- Let A, B be sets with cardinalities m and n respectively, then ∃ an Injection $f: A \to B \Leftrightarrow f$ is a bijection.
- Suppose A has cardinality n and B has cardinality m, then

❖ No. of functions from $A \to B$:= mn

❖ No. of Injections from $A \to B$:= {mP 0n ,, m m ≥ < n n

❖ No. of surjections from $A \to B$:= {∑m r=1(-10)m-r mCr ,, m m ≤ > n n

❖ No of Bijections from $A \to B$:= {n0! ,, m m = ≠ n n

• A graphical test to classify the function $y = f(x)$:

❖ f is one-one if any horizontal line intersects the graph at most once.

Imagine the graphs of $f(x) = ex$, the line $y = 0$, etc

❖ f is onto if any horizontal line intersects the graph atleast once

MONOTONICITY OF FUNCTIONS

MONOTONIC INCREASING FUNCTION

A function is said to be a monotonic increasing function if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

Example: greatest integer function, constant function, etc

STRICTLY INCREAING FUNCTION

A function f is said to be strictly increasing function if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Example: Identity function

MONOTONIC DECRESING FUNCTION

A function f is said to be monotonic decreasing function if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

Example: Constant function

STRICTLY DECRESING FUNCTION

A function f is said to be srtictly decreasing function if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Example:$f(x) = \frac{1}{x}$

LIMIT OF A FUNCTION

LIMIT OF A FUNCTION

let $f: A \to B$ be a real valued function, then f is said to have a limit $l \in \mathbb{R}$ at a point $x = x_0$

If for every $\varepsilon > 0$, there is a real number $\delta = \delta(\varepsilon)$ such that $|x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$.

NOTE

if l is the limit of f at the point x_0, then we may write $\lim_{x \to x_0} f(x) = l$.

RIGHT AND LEFT LIMITS

- Left limit of f at $x = a$: $f(a^-) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$
- Right limit of f at $x = a$: $f(a^+) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$
- f has limit at x_0 iff both $f(a^+)$ and $f(a^-)$ exist and $f(a^+) = f(a^-) = \lim_{x \to x_0} f(x)$
- $\lim_{x \to x_0} f(x)$ is unique.

PROPERTIES OF LIMITS

Let $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$ then

- $\lim_{x \to a} cf(x) = cl$
- $\lim_{x \to a} f(x)g(x) = lm$
- $\lim_{x \to a} f(x)/g(x) = l/m, m \neq 0$
- $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} f(x)^{\left(\lim_{x \to a} g(x)\right)} = l^m$

EXAMPLES

- $\lim_{x \to 0} \frac{1}{1+x} = 1$
- $\lim_{x \to 0} x \sin \frac{1}{x} = 0$
- $\lim_{x \to 0} \frac{1}{|x|} = \infty$
- $\lim_{x \to 0} |x|^n = 0, n > 0$
- $\lim_{n \to \infty} x^n = \begin{cases} 0 & |x| < 1 \\ 1 & x = 1 \\ dosen'texist, & otherwise \end{cases}$
- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to \infty} \frac{\sin x}{x} = 0$
- $\lim_{x \to 0} \frac{\tan x}{x} = 1$
- $\lim_{x \to 0} \frac{\cos x}{x} = \infty$
- $\lim_{x \to \infty} \frac{\cos x}{x} = 0$
- $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$
- $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$

- $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$
- $\lim_{x\to 0}(1+ax)^{\frac{1}{x}} = e^a$

SEQUENCE OF REAL NUMBERS

SEQUENCE OF REAL NUMBERS

A sequence of real numbers (or a sequence in $\mathbb{R}$**)** is a function defined on the set $\mathbb{N}$ of natural numbers whose range contained in $\mathbb{R}$. If $S: \mathbb{N} \to \mathbb{R}$ is a sequence, we may denote the value of S at n by $S(n) = s_n$ and the sequence by notations like$\langle s_n \rangle, (s_n), (s_n : n \in \mathbb{N})$.

EXAMPLES

- $S := ((-1)^n : n \in \mathbb{N})$
- $S := \left(\frac{1}{2n} : n \in \mathbb{N}\right) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right)$
- Constant sequences: if $b \in \mathbb{R}, B := (b, b, b, \dots)$
- Fibonacci sequence: $F := \langle f_n \rangle$, where $f_1 = 1, f_2 = 1, f_{n+2} = f_{n+1} + f_n, n = 1,2,\dots$

BOUNDS OF THE SEQUENCE

Let $\langle s_n \rangle$ be a real sequence, then

- $\langle S_n \rangle$ is bounded above $\Leftrightarrow \exists K \in \mathbb{R}$ such that $S_n \le K \ \forall n \in \mathbb{N}$.
- $\langle S_n \rangle$ is bounded below $\Leftrightarrow \exists k \in \mathbb{R}$ such that $S_n \ge k \ \forall n \in \mathbb{N}$.
- $\langle S_n \rangle$ is bounded$\Leftrightarrow \exists k, K \in \mathbb{R}$ such that $k \le s_n \le K \ \forall n \in \mathbb{N}$.

LIMIT POINT

Let$\langle s_n \rangle$ be a real sequence, $l \in \mathbb{R}$ is said to be a limit point of $\langle s_n \rangle$

If for every $\varepsilon > 0$, there is a natural number $\mathbb{N} = \mathbb{N}(\varepsilon)$ such that for all $n \ge \mathbb{N}$, the terms S_n satisfies $|S_n - l| < \varepsilon$.

NOTE

If l is the limit for the real sequence$\langle S_n \rangle$, then we may write $S_n \to l$ or $\lim_{n\to\infty} S_n = l$. In this case the sequence $\langle S_n \rangle$ is said to be converging and say $\langle S_n \rangle$converges to l.

EXAMPLE

- $\frac{1}{\mathrm{n}} \to 0$
- $2^{\frac{1}{\mathrm{n}}} \to 1$
- $2^{1-\frac{1}{\mathrm{n}}} \to 2$

RESULT

- Convergent sequences are bounded, converse need not be true

SUBSEQUENCES

Let $A = \langle a_n \rangle$ be a sequence of real numbers and $n_1 < n_2 < \cdots < n_k < \cdots$ be strictly increasing natural numbers, then the sequence $A' = \langle a_{n_k} \rangle$given by $(a_{n_1}, a_{n_2}, \dots)$ is called a subsequence of A.

EXAMPLE

$\langle \frac{1}{2n} \rangle$is a subsequence of $\langle \frac{1}{n} \rangle$.

BOLZANO WEISTRASS THEOREM

Every bounded sequences has a limit point (and thereby a convergent subsequence).

RESULT

- Set of all limit points of bounded sequences is bounded.

LIMIT INFERIOR

Let $A = \langle a_n \rangle$ be a sequence of real numbers, let $b_k = \inf\{a_k, a_{k+1}, a_{k+2}, \dots\}, k = 1,2,3 \dots$, then it is clear that $\langle b_k \rangle$ is an increasing sequence. The limit inferior of $\langle a_n \rangle$, denoted by $\underline{lim\langle a_n \rangle}$,is given by

$$\underline{lim\langle a_n \rangle} = \lim inf\langle a_n \rangle = sup\langle b_k \rangle$$

LIMIT SUPERIOR

Let $A = \langle a_n \rangle$ be a sequence of real numbers, let $b_k = \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}, k = 1,2,3, \dots$, then it is clear that $\langle b_k \rangle$ is a decreasing sequence. Then the limit superior of $\langle a_n \rangle$, denoted by $\overline{lim\langle a_n \rangle}$ is given by

$$\overline{lim\langle a_n \rangle} = \lim sup\langle a_n \rangle = inf\langle b_k \rangle$$

NOTE

Let $\langle a_n \rangle$ be a realsequence, then

- $\langle a_n \rangle$ converges to $l \Leftrightarrow \underline{lim\langle a_n \rangle} = \overline{lim\langle a_n \rangle} = l.$
- $inf\langle a_n \rangle \leq \underline{lim\langle a_n \rangle} \leq \overline{lim\langle a_n \rangle} \leq sup\langle a_n \rangle$

EXAMPLE

1. Consider the sequence $\langle a_n \rangle = 2^{1-\frac{1}{n}}$, then it is clear that $inf\langle a_n \rangle = 2^{1-\frac{1}{n}} = 1$and $\underline{lim\langle a_n \rangle} = \overline{lim\langle a_n \rangle} = sup\langle a_n \rangle = 2$
2. Consider the sequence $\langle a_n \rangle = \frac{1}{n}$, then clearly $inf\langle a_n \rangle = \underline{lim\langle a_n \rangle} = \overline{lim\langle a_n \rangle} = 0$ and $sup\langle a_n \rangle = 1$

NOTE

Suppose $\langle a_n \rangle$ be a bounded sequence then

- $\langle a_n \rangle$ is bounded $\Leftrightarrow$ $\underline{lim}\langle a_n \rangle$ and $\overline{lim}\langle a_n \rangle$are finite.
- $\underline{lim}\langle -a_n \rangle = -\overline{lim}\langle a_n \rangle$
- $\overline{lim}\langle -a_n \rangle = -\,\underline{lim}\langle a_n \rangle \Leftrightarrow -\overline{\overline{lim}\langle -a_n \rangle} = \underline{lim}\langle a_n \rangle$

Suppose that $\langle a_n \rangle, \langle b_n \rangle$ are bounded real sequence, then

- $\underline{lim}\langle a_n \rangle + \underline{lim}\langle b_n \rangle \leq \underline{lim}\langle a_n + b_n \rangle \leq \underline{lim}\langle a_n \rangle + \overline{lim}\langle b_n \rangle \leq \overline{lim}\langle a_n + b_n \rangle \leq \overline{lim}\langle a_n \rangle + \overline{lim}\langle b_n \rangle$
- $\underline{lim}\langle a_n \rangle \underline{lim}\langle b_n \rangle \leq \underline{lim}\langle a_n b_n \rangle \leq \underline{lim}\langle a_n \rangle \overline{lim}\langle b_n \rangle \leq \overline{lim}\langle a_n b_n \rangle \leq \overline{lim\langle a_n \rangle lim\langle b_n \rangle}$

TYPES OF SEQUENCES

OSCILLATING SEQUENCE

1. Finitely oscillating.
   - $\langle a_n \rangle$ is bounded but not converging
   - $\underline{lim}\langle a_n \rangle \neq \overline{lim}\langle a_n \rangle$
   - **Ex**:$a_n = (-1)^n, a_n = 1 + (-1)^n$
2. Infinitely oscillating
   - $\underline{lim}\langle a_n \rangle = -\infty$ and$\overline{lim}\langle a_n \rangle = \infty$
   - **Ex:** : $a_n = (-1)^n n, a_n = (-2)^n$

MONOTONE SEQUENCES

A real sequence $\langle a_n \rangle$ is said to be Monotone if $\langle a_n \rangle$ satisfies either $a_n \leq a_{n+1}, \forall n$ or $a_n \geq a_{n+1}, \forall n$. In first case sequence is said to be increasing and in the later case sequence is said to be decreasing.

DIVERGING SEQUENCES

Sequences having limit $\mp\infty$.

Ex: $a_n = -n, a_n = 2^n$

CAUCHY SEQUENCE

A real sequence $\langle a_n \rangle$ is said to be Cauchy if for all $\varepsilon > 0, \exists N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon, \forall n, m \geq N$.

NOTE

let $\langle a_n \rangle$ be areal sequence, then

- $\langle a_n \rangle$ is Cauchy $\Leftrightarrow$ $\langle a_n \rangle$ is convergent.
- $\langle a_n \rangle$ is Cauchy $\Rightarrow$ $\langle a_n \rangle$ is bounded.

ENTRI

CAUCHY CRITERION

- Let $\langle a_n \rangle$ be a real sequence, if for some $0 < \alpha < 1$, $|a_{n+1} - a_n| < \alpha^n, \forall n$, then $\langle a_n \rangle$ is Cauchy.
- Let $\langle a_n \rangle$ be a real sequences, if for some $0 < \alpha < 1$, $|a_{n+1} - a_n| < \alpha |a_n - a_{n-1}|, \forall n \Rightarrow \langle a_n \rangle$ is Cauchy.

RESULT

let $\langle a_n \rangle, \langle b_n \rangle$ be real sequences, then

- $|a_{n+1} - a_n| \to 0 \not\Rightarrow |a_{n+1} - a_n| < \alpha^n, \forall n$, for some $0 < \alpha < 1$.
- $|a_{n+1} - a_n| \to 0 \not\Rightarrow |a_{n+1} - a_n| <, \alpha |a_n - a_{n-1}| \forall n$, for some $0 < \alpha < 1$.

SANDWICH THEOREM (SQUEEZ THEOREM)

Let $\langle x_n \rangle, \langle y_n \rangle, \langle z_n \rangle$ be real sequence such that $x_n \le y_n \le z_n, \forall n$, then $\langle x_n \rangle, \langle z_n \rangle \to l \Rightarrow \langle y_n \rangle \to l$

CAUCHYS THEOREM ON LIMITS

CAUCHY'S FIRST THEOREM

Let $\langle S_n \rangle$ be a real sequence such that $\langle S_n \rangle \to l$, whether finite or infinite,

then $\langle \frac{s_1 + s_2 + \cdots + s_n}{n} \rangle \to l$

COROLLARY

Let $\langle S_n \rangle$ be a real sequence such that $\langle S_n \rangle \to l, S_n \ge 0$,

then $\langle (s_1 s_2 \ldots s_n)^{\frac{1}{n}} \rangle \to l$

CAUCHY'S SECOND THEOREM

Let $\langle s_n \rangle$ be a real sequence such that $\langle s_n \rangle \to l, s_n \ge 0$, then $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} (s_n)^{\frac{1}{n}} = l (\ne \infty)$.

CESARO'S THEOREM

Let $\langle a_n \rangle, \langle b_n \rangle$ be real sequence so that $\langle a_n \rangle \to a, \langle b_n \rangle \to b$, then $\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = ab$

RESULT

- Let $\langle a_n \rangle$ be a real sequence, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l > 1 \Rightarrow \lim_{n\to\infty} a_n = \infty$

MONOTONE CONVERGENT THEOREM

let $\langle a_n \rangle$ be a real monotone sequence, then $\langle a_n \rangle$ is convergent $\Leftrightarrow \langle a_n \rangle$ is bounded.

NOTE

Let $\langle a_n \rangle$ be a real sequence, then

- $\langle a_n \rangle$ is bounded and monotonically increasing $\Leftrightarrow \langle a_n \rangle$ is convergent to its sup
- $\langle a_n \rangle$ is bounded and monotonically decreasing $\Leftrightarrow \langle a_n \rangle$ is convergent to its inf

CANTOR'S NESTED INTERVAL THEOREM

Let $\langle a_n \rangle, \langle b_n \rangle$ be real sequences so that $a_n \leq b_n, \forall n$ and $\langle a_n \rangle \to a, \langle b_n \rangle \to b$ by letting $I_n = [a_n, b_n]$, suppose that $I_{n+1} \subseteq I_n \forall n$ then $\bigcap_{n=1}^{\infty} I_n = \{a\} = \{b\}$.

- ie, $\langle a_n \rangle \& \langle b_n \rangle$ converges to the same point

MONOTONE SUBSEQUENCE THEOREM

Every sequence in R has a monotone subsequence

SERIES OF REAL NUMBERS

SERIES OF REAL NUMBERS

A series of real numbers is an expression of the form $a_1 + a_2 + a_3$ ... or more compactly as $\sum_{n=1}^{\infty} a_n$,where (a_n) is a sequence of real numbers. The number a_n is called the n-th term of the series and the sequence $S_n = \sum_{i=1}^{n} a_i$ is called the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$

CONVERGENCE AND DIVERGENCE OF SERIES

A series $\sum_{n=1}^{\infty} a_n$ is said to be converge (to $S \in \mathbb{R}$) if the sequence of partial sum of the series converge $(to\ S \in \mathbb{R})$.

NOTE

- If $\sum_{n=1}^{\infty} a_n$converges to S, then we write $S = \sum_{n=1}^{\infty} a_n$
- A series which does not converge is called divergent series.
- $\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n$

CAUCHY CRITERION

The series $\sum_{n=1}^{\infty} a_n$ of real term is convergent $\Leftrightarrow \langle s_n \rangle$ is convergent.

NOTE

- The series $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \langle a_n \rangle \to 0$.
- Above implication is from $a_n = S_{n+1} - S_n$.

p-SERIES

p –series are the series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

RESULT

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent $\Leftrightarrow p > 1$.
- $a_n = \frac{1}{n} \to 0$, but $\sum_{n=1}^{\infty} a_n$ is not convergent.
- $\sum_{n=1}^{\infty} \frac{1}{p(n)}$ *is convergent.* $\Leftrightarrow \deg(p) > 1$, *where* $p(n)$ is a polynomial in n
- $\sum_{n=1}^{\infty} a_n$ *is convergent.* $\Leftrightarrow \langle S_n \rangle$ is cauchy.
- $\sum_{n=1}^{\infty} a_n = a \;\&\; \sum_{n=1}^{\infty} b_n = b \Rightarrow \sum_{n=1}^{\infty} (a_n + b_n) = a + b$

GEOMETRIC SERIES

- $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$, if $|r| < 1$
- $\sum_{n=1}^{\infty} r^n$ diverges for $|r| \geq 1$

ABSOLUTELY CONVERGENT SERIES

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers, then $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

- ❖ Absolutely convergent series converges.

RESULT

Consider the real series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} (a_n \pm b_n)$ then,

- ❖ If any two of the above converges, then the third also converges.
- ❖ If any one of the above converges, and any one diverges, then the third will be diverges.
- ❖ If any two of the above diverges, then we cannot say about third.
- Suppose that $\sum_{n=1}^{\infty} a_n = a$, then any type of series can be obtained from $\sum_{n=1}^{\infty} a_n$ by grouping terms without altering the order.
- $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$ is convergent$\Leftrightarrow \langle s_n \rangle$ is bounded above.

PRINGSHEIM'S THEOREM

Let $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$ be a real series so that (a_n) is a monotonically decreasing sequence, then,

$\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow n a_n \to 0$.

TEST FOR SERIES

COMPARISON TEST

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, be a real series so that $0 \leq a_n \leq b_n$,then,

(i) $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.
(ii) $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges.

LIMIT COMPARISON TEST

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, be real series so that $a_n \geq 0, b_n \geq 0 \;\&\; \lim_{n\to\infty} \frac{a_n}{b_n} = l(\neq 0) < \infty$, then

(i) $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converges or diverges together.

(ii) Suppose $l = 0$ & $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
(iii) Suppose $l = \infty$ & $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

CAUCHY'S ROOT TEST FOR +VE TERM SERIES

Let $\sum_{n=1}^{\infty} a_n$ be a real series so that $a_n \geq 0$, & $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = l$, then

(i) $\sum_{n=1}^{\infty} a_n$ converges if $l < 1$
(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $l > 1$
(iii) Test fails if $l = 1$

De ALEMBERT'S RATIO TEST FOR +VE TERM SERIES

Let $\sum_{n=1}^{\infty} a_n$ be a real series so that $a_n \geq 0$, & $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then

(i) $\sum_{n=1}^{\infty} a_n$ converges if $l < 1$
(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $l > 1$
(iii) Test fails if $l = 1$

- $\log n < n^c$ for any $c > 0$.

RAABE'S TEST

Let $\sum_{n=1}^{\infty} a_n$ be a real series so thet $a_n \geq 0$ & $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = l$, then

(i) If $l > 1, \sum_{n=1}^{\infty} a_n$ converges.
(ii) If $l < 1, \sum_{n=1}^{\infty} a_n$ diverges.
(iii) If $l = 1$ test fails.

LOGARITHM TEST

Let $\sum_{n=1}^{\infty} a_n$ be a real series so that $a_n \geq 0$ & $\lim_{n\to\infty} \left(\log(\frac{a_n}{a_{n+1}})\right) = l$, then

(i) If $l > 1, \sum_{n=1}^{\infty} a_n$ converges.
(ii) If $l < 1, \sum_{n=1}^{\infty} a_n$ diverges.
(iii) If $l = 1$ test fails.

CONDENSATION TEST

Let $\sum_{n=1}^{\infty} a_n$ be a real series, then $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} 2^n a_{2^n}$ behaves alike.

ALTERNATING SERIES (LEBINIZ) TEST

Let $\sum_{n=1}^{\infty} (-1)^n a_n, a_n \geq 0$ is convergent if, $a_n \geq a_{n+1}$ & $a_n \to 0$.

CONDITIONALLY CONVERGENT SERIES

A convergent series $\sum_{n=1}^{\infty} a_n$ is said to be conditionally convergent if $\sum_{n=1}^{\infty} |a_n|$ is not convergent.

ABLE'S TEST

Let $\sum_{n=1}^{\infty} a_n$ be a real convergent series and $\langle b_n \rangle$ be a positive monotone decreasing sequence, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

- Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be convergent series, then $\sum_{n=1}^{\infty} a_n\, b_n$ also convergent.
- Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent real series, $\langle b_n \rangle$ is a real bounded sequence, then$\sum_{n=1}^{\infty} a_n\, b_n$ convergent.

EXAMPLES FOR CONVERGENT SERIES

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots = \log 2$
- $\sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots = 2$
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{2}{3} = \frac{1}{1-\left(-\frac{1}{2}\right)}$
- $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1$

DIRECHLET THEOREM

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent seires, By rearrangements (Bracketing) of terms of $\sum_{n=1}^{\infty} a_n$, we can make an absolutely convergent series converges to same sum.

COROLLARY

Let $\sum_{n=1}^{\infty} a_n$ ba a divergent series of positive terms, by rearrangements of terms of $\sum_{n=1}^{\infty} a_n$ we can make only divergent series.

REIMANN THEOREM

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent real series, then by appropriate rearrangements of terms of $\sum_{n=1}^{\infty} a_n$ we can make series so that,

(i) Convergent to any number $l < \infty$
(ii) Divergent to ∞
(iii) Divergent to $-\infty$
(iv) Oscillates finitely.
(v) Oscillates infinitely

COROLLARY

Let $\sum_{n=1}^{\infty} a_n$ be aconditionally convergent real series, and $\sum_{n=1}^{\infty} a_n = 0$, denotes $S_k = \sum_{n=1}^{\infty} a_n$, then

1. $S_k = 0$, for infinitely many k
2. It is possible that $S_k > 0$, for infinitely many k.
3. It is possible that $S_k < 0$, for infinitely many k
4. It is possible that $S_k > 0$, for all but finitely many k

NOTE

m –tail of a series $\sum_{n=1}^{\infty} a_n$:

Let $\sum_{n=1}^{\infty} a_n$ be a real series, then the m-tail of the series is given by

$$\sum_{n=m}^{\infty} a_n = a_m + a_{m+1} + a_{m+2} + \cdots$$

RESULT

- $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \sum_{n=m}^{\infty} a_n \to 0\ as\ n \to \infty$
- $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} e^{a_n}$ doesn't converge.

CONTINUITY

CONTINUITY OF A FUNCTION

Let $f: A \to B$ be a function, then f is said to be continuous at the point $x = x_0 \in A$ if $\lim_{x \to x_0} f(x) = f(x_0)$. Otherwise f is said to have a discontinuity at x_0.

EXAMPLE

- $\sin\frac{1}{x}, \cos\frac{1}{x}$ are Disccontinuous at 0.
- $f(x) = x^a \sin\frac{1}{x}, a > 0$ is continuous at 0

NOTE

- f is said to be Left continuous if $f(a^-) = f(a)$.
- f is said to be right continuous if $f(a^+) = f(a)$.

TYPES OF DISCONTINUITIES

REMOVABLE DISCONTINUITY

f is said to have a removable discontinuity at $x = a$ if $f(a^-) = f(a^+) \neq f(a)$.

Discontinuity can be removed by redefining functions at a by $f(a) = f(a^-) = f(a^+)$.

NON REMOVABLE DISCONTINUITY

1st KIND: f is said to have a discontinuity of 1st kind at $x = a$ if $f(a^-) \neq f(a^+)$

Example : Signum function , $f(x) = \begin{cases} 1 & , x > 0 \\ -1 & , x < 0 \\ 0 & , x = 0 \end{cases}$

2nd KIND: f is said to have a discontinuity of 2nd kind at $x = a$ if either or both $f(a^-), f(a^+)$ doesn't exist.

NOTE

- Monotone functions doesn't have the discontinuity of 2nd kind.

- f is continuous in $[a,b] \Rightarrow f$ is bounded on $[a,b]$
  - ❖ $f(x) = \frac{1}{x-a}$ is continuous in (a,b), but not bounded.
- f continuous in $[a,b] \Rightarrow f$ attains its bounds atleast once on $[a,b]$.
- f is continuous in $[a,b] \Rightarrow R(f) = [l,L]$, where l :=Minimum of f & L :=Maximum of f.
- f is continuous at $c \in (a,b) \Rightarrow f(x)$ has same sign as $f(c)$ in some neighbourhood $(c-\delta, c+\delta)$.

LOCATION ROOT THEOREM

Let f be continuous in $[a,b]$, $f(a)$& $f(b)$ are in opposite signs, then $\exists c \in (a,b)$ such that $f(c) = 0$.

INTERMEDIATE VALUE THEOREM

Let f be continuous in $[a,b]$, $f(a) \neq f(b)$, then f assumes all values between $f(a)$& $f(b)$.

FIXED POINT THEOREM

Let $f: [a,b] \to [a,b]$ be continuous, then $\exists c \in [a,b]$ such that $f(c) = c$. (Fixed point)

EXAMPLE

(i) f is constant ⇒ f has only one fixed point.

(ii) f is the identity function ⇔all points are fixed points.

NOTES

- Let f be a function from $[a,b]$, & $\exists c \in [f(a), f(b)]$ such that $c \notin R(f) \Rightarrow f$ is not continuous.
- Let f be injective on $[a,b]$, and satisfies Intermediate Value Theorem⇒f is continuous.
- Let f be continuous in $[a,b]$, then f assumes all values between $l = \min f$ & $L = \max f$
- Let f be continuous in $[a,b]$ and is Monotonic increasing, then $R(f) = [f(a), f(b)]$.
- $f: [a,b] \to [a,b]$ has no fixed points at all ⇒ f is not continuous.
- Let f be continuous on $[a,b]$& $f(a) = f(b) \Rightarrow \exists a_i, b_i \in (a,b)$ such that $f(a_i) = f(b_i)$.

UNIFORM CONTINUITY

A function f defined on an interval I is said to be uniformly continuous on I if for each $\varepsilon > 0$ there exist a $\delta > 0$ such that, $|f(x_2 - f(x_1)| < \varepsilon$ for arbitrary points $x_1,\ x_2$ of I for which $|x_1 - x_2| < \delta$

NON-UNIFORM CONTINUITY CRITERION

Let $I \subseteq R$ and let $f: I \to R$. Then the following satatements are equivalent.

1. f is not uniformly continuous on I.
2. there exist an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δin I such that $|x_\delta - u_\delta| < \delta\ and\ |f(x_\delta - f(u_\delta)| \geq \varepsilon_0$.

3. There exist an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in I such that $lim\ (x_n - u_n) = 0$ and $|f(x_n - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

UNIFORM CONTINUITY THEOREM

Let I be a closed and bounded interval and let $f: I \to R$ be continuous on I. Then, f is uniformly continuous on I.