

## Two-Dimensional Collisions in Center-of-Mass Reference Frame

### 15.7.1 Two-Dimensional Collision in Center-of-Mass Reference Frame

Consider the elastic collision between two particles in the laboratory reference frame

(Figure 15.9). Particle 1 of mass  $m_1$  is initially moving with velocity  $\mathbf{v}_{1,i}$  and collides

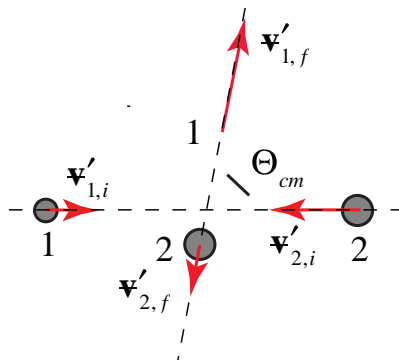
elastically with a particle particle 1 moves with velocity 2 of mass  $m_2$  that is initially at rest. After the collision the

and particle 2 moves with velocity  $\mathbf{v}_{2,f}$

15.7.1 we determined how to find  $v_{1,f}$ ,  $v_{2,f}$ , and  $\theta_{2,f}$  in terms of  $v_{1,i}$  and  $\theta_{2,f}$ . We shall now analyze the collision in the center-of-mass reference frame, which is boosted from the laboratory frame by the velocity of center-of-mass given by

$$\mathbf{v}_{cm} = \frac{m_1 \mathbf{v}_{1,i}}{m_1 + m_2} \quad (15.5.36)$$

Because we assumed that there are no external forces acting on the system, the center-of-mass velocity remains constant during the interaction.



**Figure 15.13** Two-dimensional elastic collision in center-of-mass reference frame

Recall the velocities of particles 1 and 2 in the center-of-mass frame are given by (Eq.,(15.2.9) and (15.2.10)). In the center-of-mass reference frame the velocities of the two incoming particles are in opposite directions, as are the velocities of the two outgoing particles after the collision (Figure 15.13). The angle  $\Theta_{cm}$  between the incoming and outgoing velocities is called the **center-of-mass scattering angle**.

### 15.7.2 Scattering in the Center-of-Mass Reference Frame

Consider a collision between particle 1 of mass  $m_1$  and velocity  $\mathbf{v}_{-1,i}$  and particle 2 of mass  $m_2$  at rest in the laboratory frame. Particle 1 is scattered elastically through a scattering angle  $\Theta$  in the center-of-mass frame. The center-of-mass velocity is given by

$$\mathbf{v}_{cm} = \frac{m_1 \mathbf{v}_{1,i}}{m_1 + m_2} \quad (15.5.37)$$

In the center-of-mass frame, the momentum of the system of two particles is zero

$$\mathbf{0} = m_1 \mathbf{v}'_{1,i} + m_2 \mathbf{v}'_{2,i} = m_1 \mathbf{v}'_{1,f} + m_2 \mathbf{v}'_{2,f}. \quad (15.5.38)$$

Therefore

$$\mathbf{v}'_{1,i} = -\frac{m_2}{m_1} \mathbf{v}'_{2,i}. \quad (15.5.39)$$

$$\mathbf{v}'_{1,f} = -\frac{m_2}{m_1} \mathbf{v}'_{2,f}. \quad (15.5.40)$$

The energy condition in the center-of-mass frame is

$$\frac{1}{2} m_1 v'^2_{1,i} + \frac{1}{2} m_2 v'^2_{2,i} = \frac{1}{2} m_1 v'^2_{1,f} + \frac{1}{2} m_2 v'^2_{2,f}. \quad (15.5.41)$$

Substituting Eqs. (15.5.39) and (15.5.40) into Eq. (15.5.41) yields

$$v'_{1,i} = v'_{1,f}.$$

(we are only considering magnitudes). Therefore

$$v'_{2,i} = v'_{2,f}. \quad (15.5.42)$$

Because the magnitude of the velocity of a particle in the center-of-mass reference frame is proportional to the relative velocity of the two particles, Eqs. (15.5.42) and (15.5.43) imply that the magnitude of the relative velocity also does not change

$$|\mathbf{v}_{1,2,i}| = |\mathbf{v}_{1,2,f}|, \quad (15.5.44)$$

verifying our earlier result that for an elastic collision the relative speed remains the same, (Eq. (15.2.20)). However the direction of the relative velocity is rotated by the center-of-mass scattering angle  $\Theta_{cm}$ . This generalizes the energy-momentum principle to two dimensions. Recall that the relative velocity is independent of the reference frame,

$$\mathbf{v}_{1,i} - \mathbf{v}_{2,i} = \mathbf{v}'_{1,i} - \mathbf{v}'_{2,i} \quad (15.5.45)$$

In the laboratory reference frame  $\mathbf{v}_{2,i} = \mathbf{0}$ , hence the initial relative velocity is  $\mathbf{v}'_{2,i} =$

$\mathbf{v}_{1,i}$ , and the velocities in the center-of-mass frame of the particles are then

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$$\mathbf{v}'_{1,i} = m_2 \mathbf{v}_{1,i} \quad (15.5.46) \quad \mathbf{v}_1$$

$$\mathbf{v}'_{2,i} = -m_1 \mathbf{v}_{1,i} \quad (15.5.47)$$

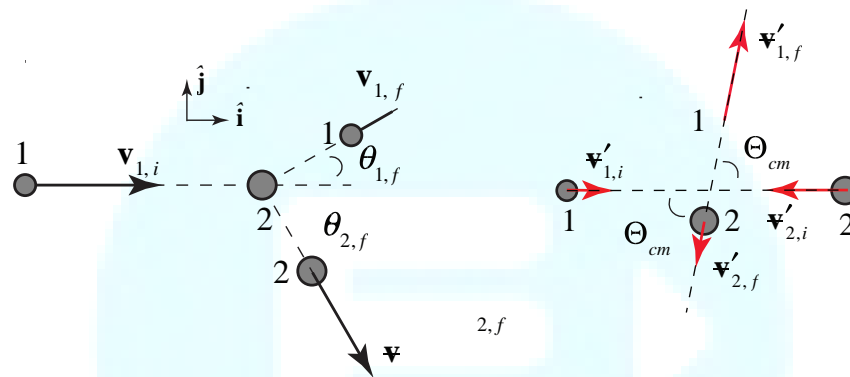
Therefore the magnitudes of the final velocities in the center-of-mass frame are

$$v'_{1,f} = v'_{1,i} = m_2 v_{1,i} \quad v'_{2,f} = v'_{2,i} = m_1 v_{1,i} \quad (15.5.48)$$

$$v'_{2,f} = \underline{v}'_{2,i} = \underline{v}_{2,i} = \underline{v}_{2,i} \quad (15.5.49) \quad v'$$

### Example 15.8 Scattering in the Lab and CM Frames

Particle 1 of mass  $m_1$  and velocity  $\mathbf{v}_{1,i}$  by a particle of mass  $m_2$  at rest in the laboratory frame is scattered elastically through a scattering angle  $\Theta$  in the center of mass frame, (Figure 15.14). Find (i) the scattering angle of the incoming particle in the laboratory frame, (ii) the magnitude of the final velocity of the incoming particle in the laboratory reference frame, and (iii) the fractional loss of kinetic energy of the incoming particle.



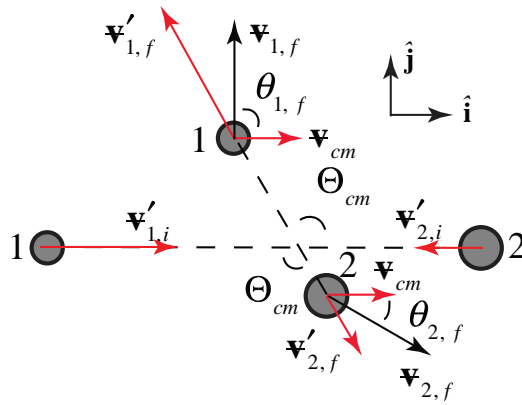
**Figure 15.14** Scattering in the laboratory and center-of-mass reference frames

#### Solution:

i) In order to determine the center-of-mass scattering angle we use the transformation law for velocities

$$\mathbf{v}_{1,f} = \mathbf{v}'_{1,f} + \mathbf{v}_{cm} \quad (15.5.50)$$

In Figure 15.15 we show the collision in the center-of-mass frame along with the laboratory frame final velocities and scattering angles.



**Figure 15.15** Final velocities of colliding particles

Vector decomposition of Eq. (15.5.50) yields

$$v_{1,f} \cos \theta_{1,i} = v'_{1,f} \cos \Theta_{cm} - v_{cm}, \quad (15.5.51)$$

$$(15.5.52)$$

$$v_{1,f} \sin \theta_{1,i} = v'_{1,f} \sin \Theta_{cm}.$$

where we choose as our directions the horizontal and vertical. Divide Eq. (15.5.52) by (15.5.51) yields

$$\tan \theta_{1,i} = \frac{v'_{1,f} \sin \Theta_{cm}}{v'_{1,f} \cos \Theta_{cm} - v_{cm}} \quad (15.5.53)$$

Because  $v'_{1,i} = v'_{1,f}$ , we can rewrite Eq. (15.5.53) as

$$\tan \theta_{1,i} = \frac{v' \sin \Theta}{v \sin \Theta - v' \sin \Theta} \quad (15.5.54)$$

$$v'_{1,i} \cos \Theta_{cm} - v_{cm}$$

We now substitute Eqs. (15.5.48) and  $v_{cm} = m_1 v_{1,i} / (m_1 + m_2)$  into Eq. (15.5.54) yielding

$$\tan\theta_{1,i} = \frac{m \sin\Theta}{\cos\Theta_2 - m_{1cm}/m_2} \quad (15.5.55)$$

Thus in the laboratory frame particle 1 scatters by an angle

$$\theta_{1,i} = \tan^{-1} \left( \frac{\cos\Theta_2 - m_{1cm}/m_2}{\sin\Theta - m_{1cm}/m_2} \right) \quad (15.5.56)$$

ii) We can calculate the square of the final velocity in the laboratory frame

$$\vec{v}_{1,f} \cdot \vec{v}_{1,f} = (\vec{v}'_{1,f} + \vec{v}_{cm}) \cdot (\vec{v}'_{1,f} + \vec{v}_{cm}) \quad (15.5.57) \text{ which becomes}$$

$$v_{1,f}^2 = v'_{1,f}{}^2 + 2\vec{v}'_{1,f} \cdot \vec{v}_{cm} + v_{cm}^2 = v'_{1,f}{}^2 + 2v'_{1,f} v_{cm} \cos\Theta_{cm} + v_{cm}^2 \quad (15.5.58)$$

We use the fact that  $v'_{1,f} = v_{1,i} (\mu/m_1)$ ,  $v_{1,2,i} = (\mu/m_1)v_{1,i} = (m_2/m_1 + m_2)v_{1,i}$  to rewrite Eq. (15.5.58) as

$$v_{1,f}^2 = \left( \frac{m_1}{m_1 + m_2} v_{1,i} \right)^2 + 2 \left( \frac{m_1}{m_1 + m_2} v_{1,i} \right) \left( \frac{m_2}{m_1 + m_2} v_{1,i} \right) \cos\Theta_{cm} + \left( \frac{m_2}{m_1 + m_2} v_{1,i} \right)^2 \quad (15.5.59)$$

Thus

$$v_{1,f} = \frac{\left( m_2^2 + 2m_2m_1 \cos\Theta_{cm} + m_1^2 \right)^{1/2}}{m_1 + m_2} v_{1,i} \quad (15.5.60)$$

(iii) The fractional change in the kinetic energy of particle 1 in the laboratory frame is given by

$$K_{1,f} - K_{1,i} = v_{1,f}^2 - v_{1,i}^2 = m_1 \left[ \frac{2m_2 v_{1,i} \cos \Theta_{cm}}{m_1 + m_2} + 2m_2 v_{cm} \cos \Theta_{cm} + m_2^2 - 1 \right] = 2m_2 m_1 (\cos \Theta_{cm} - 1) \quad (15.5.61)$$

We can also determine the scattering angle  $\Theta_{cm}$  in the center-of-mass reference frame from the scattering angle  $\theta_{1,i}$  of particle 1 in the laboratory. We now rewrite the momentum relations as

$$v_{1,f} \cos \theta_{1,i} + v_{cm} = v'_{1,f} \cos \Theta_{cm}, \quad (15.5.62) \quad v_{1,f} \sin \theta_{1,i} = v'_{1,f} \sin \Theta_{cm}. \quad (15.5.63)$$

In a similar fashion to the above argument, we have that

$$\tan \Theta_{cm} = \frac{v_{1,f} \sin \theta_{1,f}}{v_{1,f} \cos \theta_{1,f} + v_{cm}}. \quad (15.5.64)$$

Recall from our analysis of the collision in the laboratory frame that if we specify one of the four parameters  $v_{1,f}$ ,  $v_{2,f}$ ,  $\theta_{1,f}$ , or  $v_{1,i}$ , then we can solve for the other three in terms of the initial parameters  $v_{1,i}$  and  $v_{2,i}$ . With that caveat, we can use Eq. (15.5.64) to determine  $\Theta_{cm}$ .



