11. VELOCITY-DEPENDENT POTENTIAL

Let us consider the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \tilde{Q}_j \qquad j$$

— 1,3N k

for a system of N particles of mass mt and k holonomous constraints. The position ri ri (t) of each particle can be written in terms of the 3N — k generalized coordinates qj as rt rt (ql ' • • • q3N—k, t). Here we have defined the Kinetic Energy

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 ,$$

2

i=l

and the Generalized Forces

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} . \qquad (4)$$

Whenever part of these forces can be derived from a potential function V depending only on the coordinates qj, the velocities äj and, eventually, the time t, as

$$Q_j = \tilde{Q}_j + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial V}{\partial q_j} , \qquad (5)$$

the Lagrange equations can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \tilde{Q}_j ,$$

where C = T — V is the Lagrangian function. Velocitydependent potentials were introduced in

Lagrangian mechanics by the German mathematician Ernst Christian Julius Schering (1833—1897) in 1873 [10] as a way of dealing with the pre-Maxwellian Electrodynamical Theory of Wilhelm Eduard Weber (1804—1891) [11]. It was coined as Schering potential by Edmund Taylor Whittaker (1873—1956) in the first edition of his Analytical Dynamics [12], but he dropped this attribution in later editions (see [13]).

111. VELOCITY-DEPENDENT POTENTIAL OF AN ELECTROMAGNETIC FIELD

Far from being academic, the idea of a velocitydependent potential can serve as a path for solving inverse problems [14] and comprises the case of a particle of charge q in the presence of an electromagnetic field $[\mathbf{E}(\mathbf{r}, t)$,B(r, t)]. The Lorentz Force F = q(E+F x B) can be written in terms of the scalar (4) and vectorial (A) potentials (such that B = V x A and E = —V+—DA/Dt) as

$$\mathbf{F} = q\left(-\nabla\phi - \frac{\partial A}{\partial t} + \dot{\mathbf{r}} \times \nabla \times \mathbf{A}\right) . \quad (6)$$

By replacing in Eq. (4) the corresponding generalized forces can be written as in Eq.(5) when the following velocity-dependent potential is defined

$$V = q\,\phi - q\,\dot{\mathbf{r}} \times \mathbf{A} . \quad (7)$$