COMPLEX NUMBERS

General form of a complex number $z = a + ib$

The complex number $z = a\ ib$, a is called the real part, denoted by $Re\ z$ and b is called the imaginary part denoted by $Im\ z$ of the complex number z.

Example: if $z = 2\ i5$, then $Re\ z = 2$ and $Im\ z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Algebra of Complex Numbers

1. Addition of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + c) + i\,(b + d)$,

which is again a complex number.

Example:

$$(2 + i3) + (-6 + i5) = (2 - 6) + i\,(3 + 5) = -4 + i\,8$$

Properties:

(i) **The closure law :** The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law :** For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

(iii) **The associative law :** For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iv) **The existence of additive identity :** There exists the complex number 0 + i 0 (denoted as 0), called the additive identity or the zero complex number, such that, for every complex number z, $z + 0 = z$.

(v) **The existence of additive inverse:** To every complex number z = a + ib, we have the complex number – a + i(– b) (denoted as – z), called the additive inverse or negative of z.
We observe that z + (–z) = 0 (the additive identity).

2. Difference of two complex numbers

Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows: $z_1 - z_2 = z_1 + (-z_2)$.

Example:

$: (6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

$: (2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

3. Multiplication of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

Example:

$$(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2)$$

$$= -24 + i28$$

Properties:

(i) **The closure law:** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law:** For any two complex numbers z_1 and z_2, $z_1 z_2 = z_2 z_1$

(iii) **The associative law:** For any three complex numbers $z_1, z_2, z_3, (z_1 z_2)z_3 = z_1 (z_2 z_3)$.

(iv) **The existence of multiplicative identity:** There exists the complex number 1 + i 0 (denoted as 1), called the multiplicative identity such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse:** For every non-zero complex number z = a + ib or a + bi(a ≠ 0, b ≠ 0), we have the complex number

$\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the multiplicative inverse of z such that

$z\frac{1}{z} = 1$ (the multiplicative identity).

(vi) **The distributive law:** For any three complex numbers z_1, z_2, z_3,

$$(a)\ z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

$$(b)\ (z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$$

4. Division of two complex numbers

Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 . \frac{1}{z_2}$$

Example: let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

Then $\frac{z_1}{z_2} = \left((6+3i) \times \frac{1}{2-i}\right) = (6+3i)(\frac{2}{2^2+(-1)^2} + i\ \frac{-(-1)}{2^2+(-1)^2}$

$$= (6+3i)\frac{2+i}{5}$$

$$= \frac{1}{5}[12 - 3 + i(6+6)] = \frac{1}{5}(9+12i)$$

The square roots of a negative real number

Note that $\sqrt{-1} = i$

$$i^2 = -1\ and\ (-i)^2 = i^2 = -1$$

Therefore, the square roots of – 1 are i, – i.

by the symbol $\sqrt{-1}$, we would mean i only.

The equation $x^2 + 1 = 0$ has roots i and –i.

The square roots of -3 are $\sqrt{3}i$ and $\sqrt{3}i$. But the symbol $\sqrt{-3}$ is meant to represent $\sqrt{3}i$ only i.e $\sqrt{-3} = \sqrt{3}i$

Generally, if a is a positive real number $\sqrt{-a} = \sqrt{a}\sqrt{-1} = \sqrt{a}i$

Power of i

$i^2 = -1$

$i^3 = i^2 i = (-1)i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = (i^2)^2 i = (-1)^2 i = i$

ENTRI

Identities

1. $(z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$
2. $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$
3. $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$
4. $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$
5. $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

The Modulus and the Conjugate of a Complex Number

Let z = a + ib be a complex number. Then, the modulus of z, denoted by | z |, is defined to be the non-negative real number $\sqrt{a^2 + b^2}$,

i.e., $| z | = \sqrt{a^2 + b^2}$

The conjugate of z, denoted as $\bar{z}$, is the complex number a – ib, i.e., $\bar{z} = a - ib$.

Quadratic Equations

$ax^2 + bx + c = 0$ with real coefficients a, b, c and a ≠ 0.

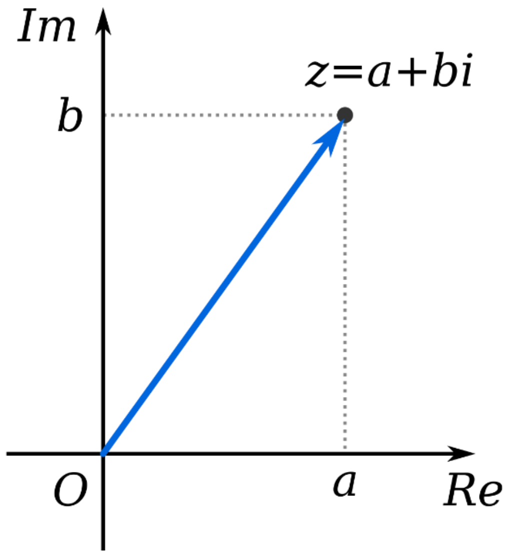
When the discriminant $b^2 4ac > 0$ we have real roots

When the discriminant $b^2 4ac > 0$ we have complex roots

Argand Plane and Polar Representation

The complex number x + iy which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point P(x, y) in the XY-plane and vice-versa.

The plane having a complex number assigned to each of its point is called the complex plane or the Argand plane.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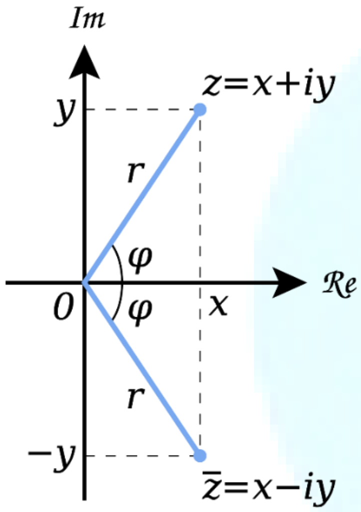
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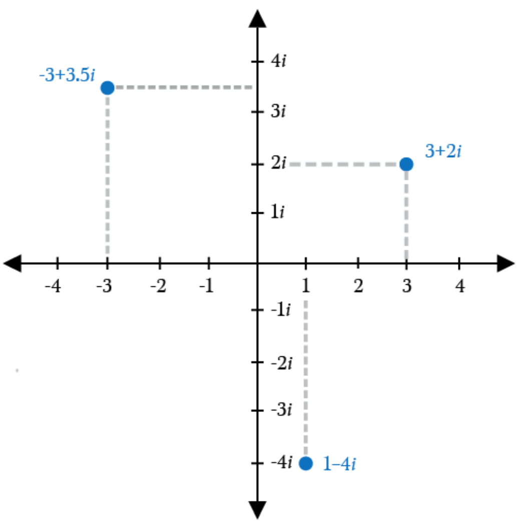
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-3+3.5i
3i
3+2i
2i
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-4
-3
-2
-1
1
2
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4
-1i
-2i
-3i
-4i
1-4i

In the Argand plane, the modulus of the complex number

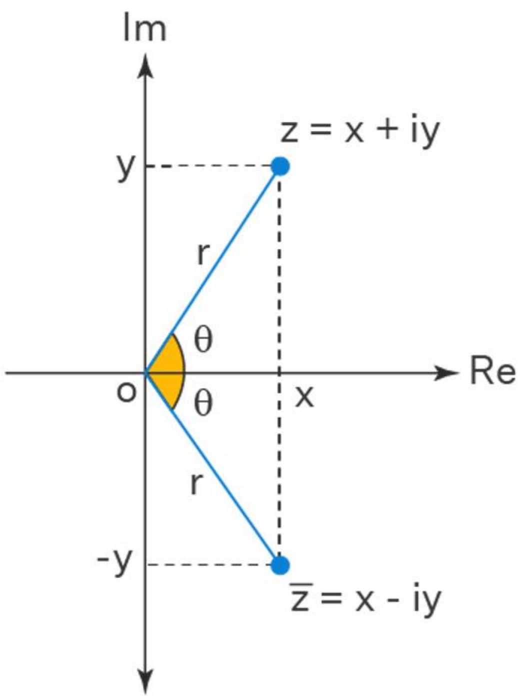
$x + iy = \sqrt{x^2 + y^2}$ is the distance between the point P(x, y) and the origin O (0, 0).

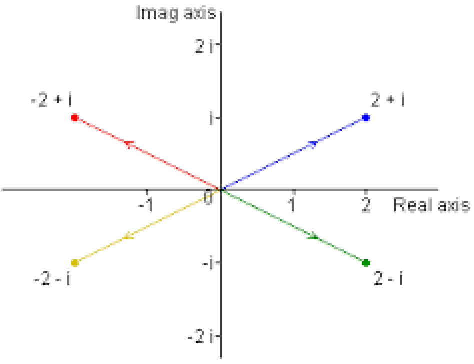
The points on the x-axis corresponds to the complex numbers of the form a + i 0 and the points on the y-axis corresponds to the complex numbers of the form 0 + i b.

The x-axis and y-axis in the Argand plane are called, respectively, the real axis and the imaginary axis.

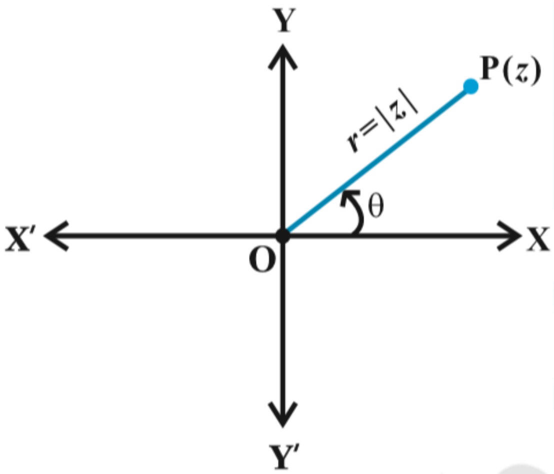
Representation of a complex number and its conugate in the argant plane:

The representation of a complex number z = x + iy and its conjugate z = x – iy in the Argand plane are, respectively, the points P (x, y) and Q (x, – y). Geometrically, the point (x, – y) is the mirror image of the point (x, y) on the real axis.





Polar representation of a complex number



Let the point P represent the nonzero complex number z = x + iy. Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis.

The point P is uniquely determined by the ordered pair of real numbers (r, θ), called the polar coordinates of the point P.

$z = x + iy$

$x = r\cos\theta, y = r\sin\theta$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is the polar form of the complex number.

$|z| = \sqrt{x^2 + y^2} = r$,is the modulus of z

θ is the argument of z.

- arg z is not unique.

- In order to specify a uniue value of arg z we restrict the interval to a lengh of 2π.
- The value of θ such that $-\pi < \theta \leq \pi$ is called principal argument of z and is denoted by Arg z.

θ can be from any other interval of length 2π.

$arg\ z = Arg\ z + 2k\pi; \qquad k = 0, \pm 1. \pm 2, \ldots$

- $arg(z_1 z_2) = arg\ z_1 + arg\ z_2$
- $arg\left(\frac{z_1}{z_2}\right) = arg\ z_1 - arg\ z_2$
- $arg\ \bar{z} = -arg\ z$
- $arg|z| = 0$
- $arg(-z) = arg(-1) + arg\ z = \pi + arg\ z$
- $arg\ \left(\frac{1}{z}\right) = \pi - arg\ z$
- $\bar{z} = a - ib$
- $\overline{z_1 z_2} = \bar{z_1}$
- $\overline{kz_2} = k\bar{z_2}\ \bar{z_2}$
- $\left(\frac{\bar{z_1}}{z_2}\right) = \frac{\bar{z_1}}{\bar{z_2}}$
- $\frac{z+\bar{z}}{2} = Re(z)$
- $\frac{z-\bar{z}}{2} = Re(z)$
- $|z| = \sqrt{x^2 + y^2}$
- $|z_1 z_2| = |z_1||z_2|$
- $|kz| = |k||z|$
- $|z| = |\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq |z_1| - |z_2|$
- $|z_1| - |z_2| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

Exponential function

- $|e^z| = e^x$
- $Arg(e^z) = y, z = x + iy, -\pi \le y < \pi$
- $y = y0 \Rightarrow e^z$ is a ray from the origin with $Arg = y0$
- $e^{z+2\pi i} = e^z$
- $\frac{e^{iz}+e^{-iz}}{2} = \cos z$
- $\frac{e^{iz}-e^{-iz}}{2i} = \sin z$
- $\frac{e^{z}-e^{-z}}{2} = \sinh z$
- $\frac{e^{z}+e^{-z}}{2} = \cosh z$
- $cos\ iz = cosh\ z$
- $sin\ iz = i\ sinh\ z$

Set of Points in the complex plane

Let S be a non-empty set of complex numbers and $z_0 \in S$ be any complex number and δ be any two real number. Then we define the following:

1. **Circle**: the set of points which satisfies the equation

$$|z - z_0| = \delta \text{ or } (x - x_0)^2 + (y - y_0)^2 = \delta^2$$

2. Open disc

An open disc is defined as

$$\Delta(z_0, \varepsilon) = \{z \in \mathbb{C}: |z - z_0| < \varepsilon\}$$

With centro z_0 and ractius ε

ie set of all points inside the circle $|z - z_0| = \varepsilon$

3. Closed disc

A closed disc is defined as $\Delta(z_0; \varepsilon) = \{z \in \mathbb{C}: |z - z_0| \le \varepsilon\}$ with centre at the point z_0 and radius ε . ie set of all points inside and on the circle $|z - z_0| = \varepsilon$

4. Annulus

The set of points which lie between two concentric circles

$C_1: |z - z_0| = r_1$ and $c_2: |z - z_0| = r_2$ defines an open annulus or an open circular ring.

ie set of points which satisfies the inequality $r_1 < (z - z_0) < r_2$ and the set of points which silesfies the inequality $r_1 \leqslant |z - z_0| \leqslant r_2$ defines a closed annulus.

5. Neighborhood of a Point

A δ nhd of a point z_0 in the complex plane is the set of all points z which lie in the open disc $|z - z_0| < \delta$.
Denoted as $N_\delta(z_0)$ or $N(z_0, \delta)$
If we exclude the point z_0 from $|z - z_0| < \delta$ then it is called deleted nhd of δ.

6. Interior point

A point z is in the interior point of S , if for some $\delta > 0, N(z, \delta) \subseteq S$

7. Exterior Point

A point z is in an exterior point of S, if it is in the interior of the set S^c.

8. Frontier Point [FrS]

$z \in \mathbb{C}$ is said to be Frontier point of $S \subseteq \mathbb{C}$ if it is neither exterior point nor interior point.

9. Boundary Point (δ S)

Frontier points of S which are member of S .

10. Open, Set

A set S is open if every point is an interior point.

$$\begin{aligned} \text{ex: } S &= \{z: |z - z_0| < r\} \\ : S &= \{z: \text{Re } z < 0\} \end{aligned}$$

11. Closed Set

A set S is closed if every boundary point of S belongs to S.
ex: $S = \{z: |z - z_0| \leq r\}$

12. Bounded Set

An open set S is bounded, if $\exists$ a +ve real number M such that
$|z| \leq M \ \forall z \in S$
otherwise it is unbounded

$$\begin{aligned} \text{ex: } S &= \{z: |z - z_0| < r\} \text{ bounded} \\ : 5 &= \{z: |z - z_0| > 0\} \text{ unbounded} \end{aligned}$$

13.Connected Set

A subset S of $\mathbb{C}$ is said to be connected if the only subsets of S which are both open and closed are ϕ and S.

14.Domain [D]

An open connected set is called domain

15.Region $[\boldsymbol{R}]$

A region is a domain together with all, some or none of is boundary points.

- A domain is always a region but a region may or may not be a domain.
  ex: An open disc is both domain and region
  : closed disc is region but not domain

16.Extended Complex plane

The complex plane to which the point at $z = \infty$ has been added.

17.finite complex plane

The complex plane without the point $z = \infty$

LIMIT CONTINUITY & DIFFERENTIABILITY

Limit of A Function

Let $w = f(z)$ be a complex valued function f defined on $D \subseteq \mathbb{C}$ let $z_0 \in D$.

Then f is said to have a limit l as $z \to z_0$ and we write

$\lim_{z \to z_0} f(z) = l$ or $f(z) \to l$ as $z \to z_0$
if and only if for any given $\varepsilon > 0, \exists a\delta > 0$ such that $|f(z) - l| < \varepsilon$ whenever $z \in D$ &

$0 < |z - z_0| < \delta$

i.e., if and only if for each $\varepsilon > 0, \exists \delta > 0$ such that $f(z) \in B(l; \varepsilon)$ whenever $z \in B(z_0, \delta) \setminus \{z_0\} \cap B$.
$z \to z_0$ in complex plane.

It is straight forward to state $\lim_{z \to z_0} f(z) = l \Leftrightarrow \lim_{z \to z_0} |f(z) - l| = 0$

Note:
(i) The function need not to be defined at z_0, in order to have a limit at z_0.
(ii) It is the punctured disk $B(z_0, \delta) \setminus \{z_0\}$ which is involved in D, i.e., z_0 need

not to be in D.

(iii) If the condition that $z_0 \in D$ holds, we may have $f(z_0) \neq l$.

Alter Definition of Limit of A Function

Let $f(z) = u(z) + iv(z)$, where $u(z) = u(x, y)$ & $v(z) = v(x, y)$ are real valued functions, be defined on D except possibly at z_0. Then for l_1 & $l_2 \in R$.

$\lim_{z \to z_0} f(z) = l_1 + il_2$ if and only if

$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = l_1$ &

$\lim_{(x,y) \to (x_0,y_0)} v(x, y) = l_2$.

- If $\lim_{z \to z_0} f(z)$ exists, then it is unique.
- Let $f(z)$ & $g(z)$ be two functions such that

  $\lim_{z \to z_0} f(z) = l_1$ & $\lim_{z \to z_0} g(z) = l_2$

  Then

  (a) $\lim_{z \to z_0} [\alpha f(z)] = \alpha l_1$ where

  (b) $\lim_{z \to z_0} [f(z) \pm g(z)] = l_1 \pm l_2$

  (c) $\lim_{z \to z_0} [f(z) \cdot g(z)] = l_1 \cdot l_2$

  (d) $\lim_{z \to z_0} \left[\frac{1}{g(z)}\right] = \frac{1}{l_2}, l_2 \neq 0$

  (e) $\lim_{z \to z_0} [f(z)/g(z)] = \frac{l_1}{l_2}, l_2 \neq 0$

Limit of a Function at $z = \infty$

The function $f(z)$ has a limit L as $z \to \infty$, if for any arbitrary small real number

$\varepsilon > 0, \exists$ a real number $\delta > 0$ such that

$|f(z) - l| < \varepsilon$ whenever $|z| > \frac{1}{\delta}$

Alternately, we substitute $z = \frac{1}{\xi}$ since $\xi \to 0$ as $z \to \infty$ we obtain

$$\lim_{z\to\infty} f(z) = \lim_{\xi\to 0} f\left(\frac{1}{\xi}\right)$$

Continuous Function

A function $f: D \to \mathbb{C}$ is continuous at $z_0 \in D$ if and only if $\lim_{z\to z_0} f(z)$ exists & equals to the functional value $f(z_0)$. We say that f is continuous on D or $f: D \to \mathbb{C}$ is continuous when f is continuous at all points of D i.e., for a given $\varepsilon > 0$, there ∃ a $\delta > 0$ such that

$|f(z) - f(z_0)| < \varepsilon$ whenever $z \in D \& |z - z_0| < \delta$ or equivalently, $f(z) \in B(f(z_0); \varepsilon)$ whenever $z \in B(z_0; s) \cap D$.

Results on Continuity

1 A function is continuous by default at all isolated points of the domain.

2 Continuous function maps connected sets to connected set.

3 Continuous function maps compact set to compact set.

4 If a continuous function is defined on a compact set then it is uniformly continuous.

5 If the function $f(x) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$, then the real functions $u(x, y)$ & $v(x, y)$ are also continuous at the point (x_0, y_0).

6 If $f(z)$ & $g(z)$ are continuous at a point z_0, then the function

$f(z) \pm g(z),\ f(z) \cdot g(z),\ |f|,\ \frac{f(z)}{g(z)}$ where $g(z_0) \neq 0$ are also continuous at z_0.

7 If $f(z)$ is continuous in a closed region S_1, then it is bounded i.e., $|f(z)| \leq M\ \forall z \in S$.

8 The function $f(z)$ is continuous at $\xi = 0$.
Composite Function: Let a function $g(z)$ be defined in the neighborhood of à point z_0 & let the image of $g(z)$ in this neighborhood be contained in a region in which $f(z)$ is defined. Then the composite function $f(g(z))$ is defined for all z in the neighborhood of the point z_0.

9 The composition of two continuous function is continuous i.e., if $f: D_1 \to D_2$ is continuous at $z_0 \in D_1$ & if $g = D_2 \to \mathbb{C}$ is continuous at $\dot{w}_0 = f(z_0)$, then gof defined by $g \circ f(z) = g(f(z))$ is continuous at z_0.

10 A function f is continuous at a point $z_0 \in D$ if and only if $f(z_0) = \lim_{n\to\infty} f(z_n)$ for every sequence $\langle z_n \rangle$ such that $z_n \in D$ for $n = 1,2, \ldots .\, \& \, z_n \to z_0$ as $n \to \infty$.

11 Let $f: D \to \mathbb{C}$ be a function. Then f is continuous on D if and only if for every open set $O \subseteq \mathbb{C} f^{-1}(O) = \{z \in D: f(z) \in O\}$ is open in D.

Uniform Continuity

A function $f(z)$ is said to be uniformly continuous in a region S if for a given real number $\varepsilon > 0$ depending only on ε such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever
$0 < |z_1 - z_2| < \delta$ where z_1 & z_2 are any two points in the region S.

- It is meaningless to talk about uniform continuity at a point.
- If $f(z)$ is uniformly continuous in a region S, then $f(z)$ is continuous in S

Differentiability

A complex function f defined on a non-empty set D is differentiable at $z_0 \in D$ if the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists & the value of the limit denoted by $f'(z_0)$ is called the derivative of f at z_0.

The function f is differentiable on D if it is differentiable at every point of D

or in terms of $\varepsilon - \delta$ notation, the limit in-equation (i) exists if and only if given $\varepsilon > 0, \exists$ a $\delta = \delta(\varepsilon, z_0) > 0$ such that $\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

- Differentiability $\Rightarrow$ Continuity

Cauchy-Riemann Equations

Let $f: G \to \mathbb{C}$ be differentiable & let $u(x, y) = \text{Re } f(z), v(x, y) = \text{Im } f(z)$ for

$z \in G$. Let us evaluate the limit $f'(z) = \lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ in two different ways. First let $h \to 0$ through real values of h for $h \neq 0 \& h$ real we get

$$\frac{f(z+h)-f(z)}{h} = \frac{f(x+h+iy)-f(x+iy)}{h}$$
$$= \frac{u(x+h,y)-u(x,y)}{h} + i\frac{v(x+h,y)-v(x,y)}{h}$$

Letting $h \to 0$ gives

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now let $h \to 0$ through purely imaginary values; i.e., for $h \neq 0$ and h real,
$\frac{f(z+ih)-f(z)}{ih} = -\frac{i(u(x,y+h)-u(x,y))}{h} + \frac{v(x,y+h)-v(x,y)}{h}$
Thus $f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$
... (ii) equating the real & imaginary parts of (i) & (ii) we get the $C-R$ equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \& \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Polar Form of $C-R$ Equation

Let $f(z) = u + iv$ is a differentiable function $\& z = re^{i\theta}$ then $C-R$ equations are given by

$$u_r = \frac{v_\theta}{r} \quad \& \quad v_r = \frac{u_\theta}{r}$$

Complex Form of $C-R$ Equation

Let $f(z) = u + iv$ be a differential function.
Let $z = x + iy = z(x, y)$

$$\& \ \bar{z} = x - iy = \bar{z}(x, y)$$
$$x = \frac{z+\bar{z}}{2}, \ y = \frac{z-\bar{z}}{2i}$$

i.e. $x = x(z, \bar{z}), y = y(z, \bar{z})$

$$\Rightarrow \frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$$

$$= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} + i \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + i \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2} u_x + \frac{1}{2} u_y + i \frac{1}{2} v_x + i \frac{1}{2} v_y$$

$$= \frac{1}{2} [(u_x - v_y) + i(u_y + v_x)]$$

Since f is differentiable, so it must satisfy the $C - R$ equation i.e.,

$$u_x = v_y \& u_y = v_x$$

$\Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$ is called $C - R$ equation in complex form.

Necessary Condition for Differentiability

Let $f(z) = u + iv$ be a function then necessary condition f_0 it be differentiable is that u_x, u_y, v_x & v_y exist & satisfy

$u_x = v_y \& u_y = -v_x$ i.e., $C - R$ equation

i.e. $\frac{\partial f}{\partial \bar{z}} = 0$

Sufficient Condition for Differentiability

Let $f: S \to \mathbb{C}$ where $S \subset \mathbb{C}$ be such that $f(z) = u + iv$ & $(x_0 + iy_0) = z_0 \in S \cap S'$ then f is differentiable if

(i) u_x, u_y, v_x, v_y exist

(ii) All the partial derivative are continuous.

(iii) Also satisfy the $C - R$ equation.

- A real valued function of a complex variable either has derivative zero or the derivative does not exist.

- If f and g are differentiable at z_0, then their sum $f + g$, difference f - g, product fg, quotient f/g (where $g(z_0) \neq 0$) and the scalar multiplication cf, are also differentiable at z_0 and

$$(f \pm g)' = f' \pm g',$$
$$(fg)' = fg' + f'g$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

$$(cf)' = cf'$$

Where c is a complex constant.

SINGULARITES OF ANALYTIC FUNCTIONS

Regular Point

Let f is defined in D and z_0 is an interior point in D. We say f is regular on z_0, or equivalently, z_0 is a regular point of f if $\exists\ \delta > 0$ such that $f'(z)$ exists

$\forall z \in |z - z_0| < \delta$

- If f is regular at z_0, then ∃a open disc around z_0 such that every point of the disc is regular point.
  $f'(z)$ Exists $\forall z \in |z - z_0| < \delta$
  Let $z_1 \in |z - z_0| < \delta$ and $|z_0 - z_1| = \delta_1, \delta_2 = \delta - \delta_1$ |
  $\delta' = \min\{\delta_1, \delta_2\}\ |z - z_1| < \delta' \subset |z - z_0| < \delta \Rightarrow z_1$ is regular point of f

- Analytic Function f is said to be analytic in D if it is regular atevery point of D.

- If f is analytic in $D \Rightarrow D$ is open.

- If it is said that f is analytic at z_0 it means f is analytic in an open disc around z_0. i.e. z_0 is a regular point.

- If f is defined on D where D is open and connected then f is analytic on D iff f is differentiable at every point of D.

- If f is analytic on $D - \{a\}$ where $a \in D$ then we say f is not analytic on a, if $\lim_{z \to a} f(z)$ either doesn't exist if exist then not equal to $f(a)$.

Zeroes of Analytic Functions

$z_0 \in \mathbb{C}$ is a simple zero of an analytic function $f(z)$,

- If $f(z_0) = 0$ but $f'(z_0) \neq 0$, z_0 is the mth order zero of $f(z)$,

- If $f(z_0) = 0$, $f'(z_0) = 0$, … , $f^{m-1}(z_0) = 0$ but $f^m(z_0) \neq 0$

Note:

- z_0 is a zero of $f(z)$ of order $m \Leftrightarrow f(z) = (z - z_0)^m g(z)$, where $g(z_0) \neq 0$
- $f(z) = f_1(z) f_2(z)$, z_0 is a m[th] order zero of $f_1(z)$ and is a n[th] order zero of $f_2(z) \Rightarrow z_0$ is a zero of $f(z)$ of order $m + n$

Singularities of analytic functions

$z_0 \in \mathbb{C}$ is a singularity of $f(z)$ if f is not analytic at z_0

- **Isolated singularity**

  $z_0 \in \mathbb{C}$ is an isolated singularity of f if $\exists \delta > 0$ such that f is analytic in $0 < |z - z_0| < \delta$

- **Non isolated singularity**

  z_0 is a non-isolated singularity of f if there are singularities in any neighbourhood of z_0.

Isolated singularities

- **Removable singularity**

A singularity $z_0 \in \mathbb{C}$ is a removable singularity of $f(z)$. If $\lim_{z\to\infty} (z - z_0) f(z) = 0$ iff $f(z)$ is bounded near z_0.

- **Poles**

A singularity $z_0 \in \mathbb{C}$ is a pole of $f(z)$ if $\lim_{z \to z_0} f(z) = \infty$ iff $|f|$ is unbounded near z_0.

- Suppose $f(z)$ has a pole at $z_0 \in \mathbb{C}$ of order m, then

$$f(z) = \frac{g(z)}{(z-z_0)^m}, m \in \mathbb{N}, g(z_0) \neq 0$$

  and g is analytic at z_0

- If f is analytic and has a zero at z_0, then $\frac{1}{f}$ has a pole at z_0 and vice versa.

- **Essential singularity**

A singularity $z_0 \in \mathbb{C}$ of f is said to be essential singularity if $\lim_{z\to z_0} f(z)$ doesn't exist ($\Leftrightarrow$ $|f(z)|$ is neither bounded nor goes to ∞ near z_0). Here $f(z)$ assumes every complex number with possibly one exception in every neighbourhood of z_0.

Singularity at infinity

$z = \infty$ is a removable singularity of $f(z) \Leftrightarrow z = 0$ is removable singularity of $f(\frac{1}{z})$.

- $z = \infty$ is the pole of order m of $f(z)$ iff $z = 0$ is a pole of order m of $F(\frac{1}{z})$.
- $\boldsymbol{z} = \infty$ is essential singularity of $\boldsymbol{f(z)}$ $\Leftrightarrow$ $\boldsymbol{z} = \mathbf{0}$ is essential singularity of $f\left(\frac{1}{z}\right)$.

Entire Functions

Functions which are analytic at every point of $\mathbb{C}$.

Example: $\sin z, e^z$

Harmonic Functions

A real values function $u = u(x, y)$ defined on a domain $D \subseteq \mathbb{C}$, is called harmonic with respect to the variables x, y if

1. It possesses continuous second order partial derivatives and
2. Satisfies the Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- Suppose $f = u + iv$ is analytic on D then u and v are harmonic.
- If u is harmonic then $u + b$ is also harmonic where b is any real number.

ENTRI

Harmonic Conjugate

Let u, v be harmonic functions and $f(z) = u + iv$ is analytic in D, then v is said to be harmonic conjugate of u.

- Harmonic conjugate need not be unique.
- Suppose v and v' are teo harmonic conjugate of u then their difference is a real constant.
- Suppose u is harmonic conjugate of v and v is harmonic conjugate of u on a domain D then both u and v are constant in that domain.
- If v is harmonic conjugate of u then$-u$ is harmonic conjugate of v.
- u is harmonic in a simply connected domain, then u always posses a harmonic conjugate.

Mine Thomson's Method

Let u be defined on a simply conncected domain D, then there exist an anlytic function f in that domain D whose real part is u and imaginary part of f is the harmonic conjugate of u.

f is given by

$$f(z) = \int (u_x(z,0) - iu_y(z,0))dz$$

- Linear combination of harmonic function is harmonic.
- Product of harmonic function need not be harmonic.
- Suppose u is harmonic and v is a harmonic conjugate of u, then uv is harmonic in that domain.

Maximum minimum theorem for Harmonic function

Let U be harmonic in a domain D, then Ucannot attain its maximum or minimum unless U is constant.

- If U attains its maximum or minimum, then U is constant.
- A non-constant harmonic function in domain D cannot attain its maximum or minimum.
- Suppose U is harmonic in a bounded domain D, and on its boundary then Uattains its maximum and minimum on its boundary. If it attains its maximum or minimum in D then U is constant.

ENTRI

Power Series

Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$, $z, z_0 \in \mathbb{C}$, then we call the series

$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_{1(z-z_0)} + a_2(z-z_0)^2 + \cdots$ a power series centered at z_0, a_n's are the coefficients of the series

Radius of convergence of a P.S $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

Corresponding to a power series there exist an R with $0 < R \leq \infty$ such that the series

(i) Converges for all z such that $|z-z_0| < R$

(ii) Diverges for all z such that $|z-z_0| > R$

Then R is called radius of convergence of the power series.

$$Radius\ of\ convergence(R.C) = \frac{1}{\text{limsup } |a_n|^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \frac{1}{lim\ sup\ |a_n|^{\frac{1}{n}}}$$

$$= \frac{1}{lim|a_n|^{\frac{1}{n}}}, \text{ if } lim|a_n|^{\frac{1}{n}} \text{ exist}$$

Note:

- $\sum_{n=0}^{\infty} a_n(z-z_0)^n, \sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$ have same R.C.
- For $a_n = \frac{1}{k^n}$, $R.C = k$
- If $|z_1| = |z_2|$ with $\sum_{n=0}^{\infty} a_n z_1^n$ converges and $\sum_{n=0}^{\infty} a_n z_2^n$ diverges, then the R.C of $\sum_{n=0}^{\infty} a_n z^n$ is $|z_1|$.
- If $\sum_{n=0}^{\infty} a_n$ converges and $\sum_{n=0}^{\infty} |a_n|$ diverges, then the R.C of $\sum_{n=0}^{\infty} a_n z^n\ is\ 1.$

  Ex: $a_n = \frac{(-1)^n}{n}$

- If $\sum_{n=0}^{\infty} a_n z^n$ converges at an unbounded sequence of points, then $R.C = \infty$.
- $\langle |a_n| \rangle$ is a bounded sequence, then R.C of $\sum_{n=0}^{\infty} a_n z^n$ is ≥ 1.

Note:

- Suppose that R.C of $\sum_{n=0}^{\infty} a_n z^n$ is R, then

$$R.C\ of\ \sum_{n=0}^{\infty} a_n z^{2n} = R^{\frac{1}{2}}$$

$$R.C\ of\ \sum_{n=0}^{\infty} a_n z^{kn} = R^{\frac{1}{k}}$$

$$R.C\ of\ \sum_{n=0}^{\infty} a_n n^k z^n = R$$

$$R.C\ of\ \sum_{n=0}^{\infty} \frac{a_n}{n^k} z^n = R$$

$$R.C\ of\ \sum_{n=0}^{\infty} a_n^k z^n = R^k$$

$$R.C\ of\ \sum_{n=0}^{\infty} a_n n!\, z^n = 0$$

$$R.C\ of\ \sum_{n=0}^{\infty} \frac{a_n}{n} z^n = \infty$$

- Suppose that $R_1 = R.c\ of\ \sum_{n=0}^{\infty} a_n z^n$ and $R_2 = R.C\ of\ \sum_{n=0}^{\infty} a_n z^n$, then

$$R.C\ of\ \sum_{n=0}^{\infty} (a_n \pm b_n) z^n \geq \min\{R_1, R_2\}$$

$$R.C\ of\ \sum_{n=0}^{\infty} (a_n b_n) z^n \geq R_1 R_2$$

$$R.C\ of\ \sum_{n=0}^{\infty} \frac{a_n}{b_n} z^n \geq \frac{R_1}{R_2}$$

Differentiability of Power Series

Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R$, then the function $f(z)$ is analytic (thus differentiable) on the disk $|z - z_0| < R$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

Conversely if $f(z)$ is analytic at $z_0 \in \mathbb{C}$, then there is a unique power series representation for $f(z)$ centered at z_0 (Taylor's Series).

Note:

- From the above result, if $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R$, then the function $f(z)$ is infinitly differentiable(Holomorphic) at z_0.
- In complex analysis, Analytic functions are also called Holomorphic (Regular) functions.
- In real analysis, $f(z)$ is said to be analytic at x_0, if there is a power series expansion for f centered at x_0, but f is said to be holomorphic at x_0, if it is infinitely differentiable at x_0.

- Consider power series $\sum a_n z^n$

  If a_n assumes finite no. of different values and atleast one value repeats infinitely many times thin $R \cdot C = 1$

- Let $R \cdot c$ of $\Sigma a_n z^n = R \cdot c$ of $\Sigma \frac{1}{a_n} z^n = R$ and if $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exist then $R = 1$
- If $R \cdot C$ of $s a_n z^n = R \cdot C$ of $\Sigma \frac{1}{a_n} z^n = R$ then $R \leqslant 1$

- R.C of $\Sigma a_n z^n = R_1$ and $R \cdot C$ of $\sum b_n z^n = R_2$ then
  R.C of $\Sigma(a_n + b_n)z^n = R$

$$R = \begin{cases} \min\{R_1, R_2\} & \text{if } R_1 \neq R_2 \\ \geq R_1 & \text{if } R_1 = R_2 \end{cases}$$

- R.C is the shortest distance from the centre to the nearest singular point.

Complex Integration

Path or Curve

Any continuous $f^n: [a, b] \to \mathbb{C}$ Es called curve or path

Closed curve

A curve $\gamma: [a, b] \to \mathbb{C}$ is called closed curve of $\gamma(a) = \gamma(b)$

Simple closed curve

A closed curve $\gamma: [a, b) \to \mathbb{C}$ is simple cosed curve if γ is 1-1 in $[a, b)$ and $\gamma(a) = \gamma(b)$

Contour or smooth curve

A curve $\gamma: [a, b] \rightarrow \mathbb{C}$ is suid to be smooth if γ' exist and t γ' is continuous

Jordan Cure theorem

Every simple closed curve $\boldsymbol{\gamma}$ divide the entire complex plane into two region, one is bounded and other is unbounded.
The bounded region is called interior of the curve and the unbounded region is the exterior of the curve and the curve itself is called boundary of the two region.

Orientation of curve

Orientation is determined w.r.t the region of the curve.
If γ is the curve which the boundary of two regions A and B w.r.t the region B, γ have +ve orientation ($\because B$ lies on the left of γ) and w.r.t the region A γ have -ve orientation.

A curve γ have +ve orientation of the region bounded by γ lie to the left of γ

Complex line Integral

$\gamma: [a, b] \rightarrow \mathbb{C}$ is a contour wifh $\gamma([a, b]) \subset D$
the complex line integral

$$\int_{\gamma} f(z)dz = \int_{a}^{b} f(\gamma(t))\gamma'(t)dt$$

Properties

$$\int_{\gamma} f(z)dz = -\int_{-\gamma} f(z)dz$$

$$\int_{\gamma} [\alpha f(z) + g(z)]dz = \alpha \int_{\gamma} f(z)dz + \int_{\gamma} g(z)dz$$

- $L = L(\gamma)$ is the length of the curve and $M = \max_{t\in[a,b]} | f(\gamma(t) |$, then

$$\left|\int_{\gamma} f(z)dz\right| \leqslant ML$$

$\rightarrow ML -$ inequality

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz \quad ; \ \gamma_1(b) = \gamma_2(a)$$

- if $f = u + iv$ is analytic in an open set D containing a contour γ with parametric interval $[a, b]$ i.e $\gamma([a, b]) \subset D$ then

$$\int_{\gamma} f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

Cauchy weak theorem

if f is analytic with f' continuous inside and on a simple closed contour γ then

$$\int_{\gamma} f(x)dz = 0$$

Winding number

Let γ be a closed contour in $\mathbb{C}$ that avoids a point $a \in \mathbb{C}$. The index (or winding number) of γ about a, denoted by $n(\gamma; a)$ is given by the integral

$$n(\gamma; a) = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$$

- $n(\gamma; a)$ es an enteger
- If γ is a closed contour in $\mathbb{C}$, then the mapping $\xi \rightarrow n(\gamma, \xi)$ is a continuous function on ξ at any point $\xi \notin \gamma$.
- $\eta(\gamma, a) = 0$ for unbounded component of the closed contour γ .

Cauchy Goursat theorem

f be an analytic in an open set $D \subset \mathbb{C}$ and let z_1, z_2, z_3 be in D. Assume that the closed triangle T with vertices z_1, z_2, z_3 is contained in D. Then

$$\int_{\partial T} f(x) \cdot dz = 0$$

Cauchy's theorem

If a function $f(z)$ is analytic and continuous inside and on a simple closed contour C then

$$\int_c f(c)dz = 0$$

Cauchy's integral formula

If $f(z)$ is analytic within and on a closed contour C and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{z-a}$$

Cauchy's integral formula [Extended]

If f is analytic inside and on a simple closed curve γ and 'a' is a point inside γ then

$$\frac{1}{2\pi i}\int_{\gamma}^{0} \frac{f(z)}{(z-a)^{n+1}}dz = \frac{f^n(a)}{n!}$$

where γ has +ve orientation w.r.t 'a'

- If f is analytic on a simply connected domain D and γ is a closed curve in D. Suppose a is a point in D not on γ,then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}dz = \frac{f^n(a)}{n!}n(\gamma, a)$$

where $n(\gamma, a)$ is the winding no. of $'a'$ taken in the +ve direction

Morera's Theorem

If f iscotinuous on a domain D and $\int_{\gamma} f(z)\,dz = 0$ for all simple closed curve γ in D then f is analytic in D.

- On a simply connected domain converse of Morera's theorem is true

Taylor's Theorem

suppose f is analytic on a disc centered at a and radius R $(|z-a| \leq R)$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \ \forall\ z\ |z-a| < R$$

$$= \frac{1}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{n+1}} dz$$

where $a_n = \frac{f^n(a)}{n!}$

Cauchy's Inequality

$f(z)$ is analytic ona disc centered at 'a' and radius r and $M =$ sup or max $\{|f(z)|: |z-a| = \mathrm{R}\}$

$$= max\{|f(z)|: |z-a| \leq R\}$$

then $|f^n(a)| \leq \frac{M \cdot n!}{\gamma^n}$

Liovilli's theorem

A bounded entire function is constant

Picard's theorem

A non-constant entire f assume all complex values except possible for one point.

i.e image of a non-constant entire function is dense in $\mathbb{C}$

Identity theorem

Two entire f^{ns} on D agree on an uncountable set then they are identical.

- Two entire functions agree on a bounded infinite set then they are identical
- Two entire functions agree on an infinite set need not be identical
- Two entire functions on D agree on an infinite set and has limit point in D then they are identical.

Maximum modulus principle

If f is analytic on a domain D, thin $|f|$ cannot attains its maximum unless it is constant

- If f is a non-constant f^n on a domain D, thin if cannot attain a max on D
- If f is analytic on a domain D and and its boundary ∂D, then $|f|$ altains ifs max on ∂D
- If |f| attains its max at a point D then f is a constant.

Minimum Modulus Principle

Let f be a non-constant analytic f^n on a domain D. If f never vanish on D, then $|f|$ cannot attain its minimum in D

- If f analytic on a bdd domain D and its boundary ∂D. If f never vanish on D then $|f|$ attains its minimum ∂D and if f attains its $\min$ in D then f is constant.

Schwartz Lemma

Let $f: \Delta \rightarrow \bar{\Delta}$ or Δ (open unit disc) be analytic such that $f(0)=0$ oof order n. Then

1. $|f(z)| \leq |z|^n \leq |z| \forall z \in \Delta$
2. $|f^{\mathrm{n}}(0)| \leq n!$

The equality holds either in (1) for non-zero z in Δ or in (2) iff $f(z)=\varepsilon z^n$ where $|\varepsilon|=1$

Schwartz Pick Lemma

Let $f: \Delta \rightarrow \bar{\Delta}$ be analytic. Then

(1) $|f'(a)| \leqslant \frac{1-|f(a)|^2}{1-|a|^2} \quad \forall a \in \Delta$

(2) $\delta(ka), f(b)) \leqslant \delta(a, b) \quad \forall a, b \in \Delta$

where $\delta(z, \omega)=\frac{|z-\omega|}{|1-z\bar{\omega}|}$

The equality occur either in (1) for some a $\varepsilon\Delta$ or in (2) for some

$a, b \in \Delta, a \neq b$ iff f is an automorphism on Δ (ie an analytic bijection on $\triangle$)
ie f has the form $f(z) = e^{i\alpha}\left(\frac{z-z_0}{1-z\bar{z}_0}\right)$

Laurent Series Expansion

If f is analytic the annulus $R_1 < |z| < R_2 (O \leq R_1 < R_2 \leqslant \infty)$ then f bas a unique represcntation

$$\begin{aligned} f(z) &= \sum_{n\in\mathbb{Z}} a_n z^n \\ a_n &= \frac{1}{2\pi i}\int_c \frac{f(\xi)}{\xi^{n+1}} d\xi \; n \in Z \\ &\text{with } c = \{\xi : |\xi| = r\} \text{ and } R_1 < z < R_2 \end{aligned}$$

- If f analytic in the annulus
  $R_1 < |z-a| < R_2$ (whxre $R_1 \geqslant 0$) then for any z in the annulus, f has a unique representation

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n (z-a)^n \\ a_n &= \frac{1}{2\pi i}\int_{|\xi-a|=r} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi \qquad n \in z \end{aligned}$$

- Laurent series is expanded in a nbd of isolated singularity
- $\sum_{n=1}^{\infty} b_n (z-z_0)^n$

  the part containing -ve powers is a called principal part

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

  The part containg +ve powers is called regular part

- In the laurent series , let $r \to 0$. Then $f(x)$ is analytic in
  $|z-z_0| < R$ except at the point $z = z_0$.
- If $f(z)$ is analygic at z_0 also, then the Laurent series is same as Taylor series.
- If $|f(z)| \leq M$ it z in the annulus $r < |z-z_0| < R$ ithin

$$|a_n| \leq \frac{M}{p^n} \quad n = 0, \pm 1, \pm 2.$$

$$\forall p\ ; \quad r < p < R.$$

- Residue of $f(z)$ at $z = z_0$ is the co-efficient of $\frac{1}{z-z_0}$ in its Laurent Series expansion.
  ex: Residue of $\sin\ z$ at $z = 0$ is $z = 0$
- Entire fuunction has residue zero
  ex: $f(z) = \frac{1}{z}$
  Laurant series expansion of $1/z = \sum_{n \geqslant 0} \frac{(-1)^n (z-a)^n}{a^{n+r}}$
  ex: $f(x) = z + \frac{1}{1-z}$

$$= z + \left(1 + z + \frac{z^2}{2} + \cdots\right)$$

is t theTaylor series expansion of $f(z)$ near $z = 0$.

Analyses of singularities through Laurent series

Let $f(x)$ be a f^n having isolated singularity at $z = z_0$ then we have

$$f(x) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

Regular part principal part

Converges uniformly in $0 < |z - z_0| < R$ to $f(x)$ then

- If there is no principal part (i.e $b_n = 0$) Then $z = z_0$ is either removable singularity or regular point of $f(x)$
- If the principal part of the Laurent's expansion contain the finite number of terms thin $z = z_0$ is the pole and the highest power of $\frac{1}{z-z_0}$ is defined as the order of the pole.
- If the principal part of the Laurent's expansion contains infinite no. of terms then $z = z_0$ is essential singularity
- If $f(x)$ has pole of ordes m at $z = z_0$ then there exist $\phi(z)$ analytic and non-vanishing in $|z - z_0| < \delta$ for som $\delta > \mathrm{o}$

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

conversly if $f(x) = \frac{\phi(z)}{(z-z_0)^m}, \phi(z) \neq 0$. and $\phi(z)$ is analytic in $|z - z_0| < \delta$

$\Rightarrow f(z)$ thas pole of order m at $z = z_0$

$ex\text{:}\ f(z) = \frac{\sin\ z}{z^8}$

Meromorphic Function

A f^n defined in C is said to be meromorphic at any point $z_0 \in \mathbb{C}$ either $f(x)$is regular or z_0 is non essential singularity

- Entire f^{ns} are meromorphic not conversely
- Infinity is the only possible point for a meromorphic f^n to have essential singularity
- The function is said to be meromorphic at $z = \infty$ if $g(z) = f(1/z)$ is meromorphiec An a nbel o $z = 0$
- Sums and products of meromorphic functions are meromorphic
- The quotient of a meromorphic function is meromorphic, provided that the denominator term is not identically zero

Picard's Little theorem

Every non-constant entire f^n omits at most one complex number as its value.

- An entire f^n omits two values, then it is constant

Picard's Great Theorem

Suppose f is analytic in $D = \Delta(z_0{:}\,r)(z_0)$ and $z = z_0$, is an essential singularily of f,Then $C \mid f(D)$ is a singleton set

Cassorati-Weirstrass Theorem

If f has an essential singulardly at z_0 and w_0 is a given finite complex number, then there exist a sequence $\{z_n\}$ with $z_n \to z_0$ such that

$$f(z_n) \to w_0$$

ENTRI

CALCULUS OF RESIDUES

Residue at A Finite Point

If f has an isolated singularity at z_0, then the residue of $f(z)$ at z_0 is

$\text{Res}\,[f(z); z_0] = \frac{1}{2\pi i}\int_C f(z)dz$

Where C is any circle centered at z_0 & lying inside a disk about z_0.

- Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-h}$ bé-the Laurent's expansion of $f(z)$ near $z = z_0$ then
  $\text{Res}\,[f(z); z_0] = b_1 = \frac{1}{2\pi i}\int_C f(z)dz$ i.e. the coefficient of $\frac{1}{z-z_0}$ in Laurent's expansion.

- If f is analytic at z_0, then $\text{Res}\,[f(z), z_0] = 0$
- If f has a removable singularity at z_0, , then we have $\text{Res}\,[f(z); z_0] = 0$. In particular, if C is a simple closed contour containing only removable singularities at $z_k (k = 1,2, \ldots, n)$ inside C, then $\int_C f(z)dz = 0$.
- If f has a pole of order n at z_0, then
  $$\text{Res}\,[f(z); z_0] = \lim_{z\to z_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}((z-z_0)^n f(z))$$
- If f has a simple pole at $z = z_0$ and if h is analytic at z_0 and with $h(z_0) \neq 0$, then $\text{Res}\,[f(z)h(z); z_0] = h(z_0)\text{Res}\,[f(z); z_0]$.
- Suppose ϕ is analytic at z_0 with $\phi(z_0) \neq 0$ and g has a simple zero at z_0. Then $\text{Res}\left[\frac{\phi(z)}{g(z)}; z_0\right] = \frac{\phi(z_0)}{g'(z_0)}$.

In particular, $\text{Re s}\left[\frac{1}{g(z)}; z_0\right] = \frac{1}{g'(z_0)}$

- Suppose ϕ is analytic at z_0 with $\phi(z_0) \neq 0, g$ has a pole of order two at z_0 and h has a zero of order two at z_0. Then we have
  $\text{Res}\,[\phi(z)g(z); z_0] = \phi'(z_0)\text{Res}\,[(z-z_0)g(z); z_0]$
  $+\phi(z_0)\text{Res}\,[g(z); z_0]$

ENTRI

Residue at the Point at Infinity

Let $z=\infty$ be an isolated singular point of $f(z)$, then Residue of $f(z)$ at $z=\infty$ is given by

$\text{Res}\,[f(z);\infty]=-\text{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right]$

Cauchy Residue Theorem

If f is analytic in a domain D except for isolated singularities at $a_1, a_2 \dots a_n$, then, for any closed contour γ in D on which-none of points a_k lie, we have

$$\int_\gamma f(z)dz=2\pi i\sum_{k=1}^{n} n(\gamma;a_k)\text{Res}\,[f(z);a_k]$$

Residue Formula

If f is analytic in a domain D except for tsolated singularitie $a_1, a_2, \dots, a_n$, then for any simple closed contour ⟩ in D on which nor the points a_k lie, we have

$$\int_r f(z)dz=2\pi i\sum \text{Res}\,[f(z);a_k]$$

Here the sum is taken over all $a_k's$ inside r.

$f(z)$ can have only a finite number of singularities, because otherwise singularities of $f(z)$ would have a limit point G possibly a point at infinity & so G would not be an isolated singularity of f contrary to our assumption.

Extended Residue Formula

Let f be analytic in $\mathbb{C}$ except for isolated singularities at $a_1, a_2, \dots a_n$ Then we have

(i) The sum of all residues (including the residue at infinity) of zero.

$\text{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right]=\sum_{k=1}^{n}\text{Res}\,[f(z);a_k]$

(ii) If γ is a simple closed contour in $\mathbb{C}$ such that all a_k ' s are int to γ, then

$$\int_\gamma f(z)dz=2\pi i\text{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right].$$

Argument principle

If $f(z)$ is analytic inside and on a simple closed cure γ except for a finite no. of poles inside γ and f never vauish on γ then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{f(x)}dz = z(f) - P(f)$$

where γ has two orientation w.r.t interior
$Z(F) \rightarrow$ zeros of f including multiplicity
$P(f) \rightarrow$ poles of f inclucting order.
ex: $f(z) = \begin{cases} 1/z & z = 0 \\ 0 & z = 0. \end{cases}$ $\quad \gamma = e^{2\pi it}$

Rouche's Theorem

Let f and g are analytic inside a simple closed curve γ and $|f(z)| < |g(z)|$ on γ. Then no. of zeros of $f + g$ and g are same inside and on γ.g does not vanish on γ

Conformal Mapping/Conformality

A function $f(z)$ is said to be conformal at $z_0 \in D$ (Domain) if, whenever γ_1 & γ_2 are two parameterized curves intersecting at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$ with non-zero tangents, then the following holds:
(i) The two transformed curves $\Gamma_1 = fo\gamma_1 \& \Gamma_2 = fo\gamma_2$ have non-zero tangents at t_0.
(ii) The angle from $\Gamma_1'(t_0) = (\text{fo}\,\gamma_1)'(t_0)$ to $\Gamma_2'(t_0) = (\text{fo}\,\gamma_2)'(t_0)$ is the same as the angle from $\gamma_1'(t_0)$ to $\gamma_2'(t_0)$.

If it is conformal at each point of D then we say f is conformal in D
OR
A function $w = f(z)$ is said to be conformal at z_0 if curve in the z-plane passing through z_0& image curve in w-plane passing $f(z_0)$ preserves the angle in magnitude & sense of rotation (orientation) of angle.

Isogonal: A function that preserves the magnitude-(size)-of the angle but not sense is said to be isogonal.
Example: Find the image of the region $y = x$.

Solution: Since $f(z) = z^2\ \forall z \in \mathbb{C}$
Let $z = x + iy \in \mathbb{C}$

$$\Rightarrow f(z) = z^2 = x^2 - y^2 + i(2xy)$$

Let $f(z) = u + iv$
But $f(z) = 0 + i(2x^2)$ on the curve $y = x$
Example: Find the image of the region $y = x$ under the map $w = f(z) = \bar{z}$.
Solution: Let $w = f(z) = u + iv$ be a function
Such that $w = \bar{z} = x - iy$

$$\Rightarrow u = x \& v = -y$$

$\Rightarrow u + v = 0$ Hence under $w = f(z)$ the line $y = x$ in z - plane is mapped into a line $u + v = 0$ in w-plane i.e.

Linear Fractional/Bilinear/Mobius Transformation

The map
$w = \frac{az+b}{cz+d},\ ad - bc \neq 0$ ......(i)
Where $a, b, c \& d$ are complex constants, is called a linear fractional/Bilinear transformation, or Mobius transformation

Above equation (i) can also be writen as
$Azw + bz + cz + D = 0,\ (AD - BC \neq 0)$.........(ii)

and conversely any equation of the type (ii) can be put in the form (i) since equation (ii) is linear in z & w, or bilinear in $z \& w$ so cal bilinear transformation.

- A mobius transformation is simply a composition of one, some or of the following special types of transformation.

  (i) Translation: It is a map of the form $z \mapsto z + \alpha, \alpha \in \mathbb{C} \setminus \{0\}$. $\alpha = 0$ then it is an identity.

  (ii) Magnification or Contraction: It is a map of the fo $z \mapsto r, r \in R - \{0\}$. For $r = 1$, this is the identity map & for $r = 0$ is a constant map.

Case (i) When $r > 1$, then this is a "magnification".
Case (ii) When $r < 1$, then this is a contraction map.

(iii) Rotation: It is a map of the form $z \mapsto e^{i0}z, \theta \in \mathbb{R}^{\cdot}$. This n produces a rotation through an angle about the origin with posit sense, $\theta > 0$.
Note: The rotation coupled with magnification is referred to Dilation: $z \mapsto az(a \neq 0)$.

(iv) Inversion: It is a map of the form $z \mapsto \frac{1}{z}$ which produces geometric inversion (or reciprocal map or the inversion map.)

- The mobius transformation $T(z)$ is analytic on $C \setminus \{d/c\}$.
- If $c = 0$ then $T(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$ reduces $T(z) = \frac{a}{d}z + \frac{b}{d} = \alpha z + \beta(ad \neq 0, \alpha \neq 0)$ & called a linear map.