ENTRI

Module 1

LINEAR ALGEBRA

Vector spaces, Subspaces, Linear Independence, Basis, Dimension, Algebra of linear transformations, Algebra of matrices, Rank and determinant of matrices, Linear equations, Eigen values and eigen vectors, Cayley-Hamilton theorem, Matrix representation of linear transformations, Change of basis, Canonical forms, Diagonal forms, Triangular forms, Rational forms, Jordan forms, Inner product spaces, Orthonormal basis, Quadratic forms.

ENTRI

VECTOR SPACES

A non-empty set V is said to be a vector space over a scalar field $\mathbb{F}$ together with operations, addition and scalar multiplication, if it satisfies the following axioms:

1. If $x, y \in V$, then $x + y \in V$
2. $(x + y) + z = x + (y + z)$ for every $x, y, z \in V$ (associativity)
3. There exist $0 \in V$ such that $x + 0 = x$ for every $x \in V$ (existence of additive identity)
4. For every $x \in V$ there exist $-x \in V$ such that $x + (-x) = 0$ (existence of additive inverses)
5. $x + y = y + x$ for every $x, y \in V$ (commutativity)
6. If $c \in \mathbb{F}$ and $x \in V$, then $cx \in V$
7. $c(x + y) = cx + cy$ for every $c \in \mathbb{F}$ and every $x, y \in V$
8. $(c + d)x = cx + dx$ for every $c, d \in \mathbb{F}$ and every $x \in V$
9. $(cd)x = c(dx)$ for every $c, d \in \mathbb{F}$ and every $x \in V$;
10. $1x = x$ for every $x \in V$

EXAMPLES

- Zero space or trivial space $V = \{0\}$ over any field
- $V = F$ over any field $\mathbb{F}$
- $V = \mathbb{F}$ over any sub-field of $\mathbb{F}$
- $V = \mathbb{F}^n$ over $\mathbb{F}$ ($\mathbb{Z}_p^n$ over $\mathbb{Z}_p$)
- Polynomial Space $P(X)$ over $\mathbb{F}$, $P(X) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots \ |a_i \in \mathbb{F}\}$
- Polynomial Space $P_n(X)$ over $\mathbb{F}$,
  $P_n(X) = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \mid a_i \in \mathbb{F} \,\&\, 0 \leq m \leq n\}$
- Function space $F(X)$ over $\mathbb{F}$, $(X \neq \phi)$, $F(X) = \{f \mid f : X \to \mathbb{F}\}$
  Example; $F[0,1], C[0,1]$ over $\mathbb{R}$

SUBSPACES

A subset W of a vector space V over $\mathbb{F}$ is said to be a subspace if W is itself a vector space over the field $\mathbb{F}$.

EXAMPLE

- The polynomial space $P_n(X)$ is a subspace of Polynomial subspace $P(X)$ over $\mathbb{F}$
- The polynomial space $P(X)$ is a subspace of function space $F(X)$ over $\mathbb{F}$
- The set of all even functions on the set X form a subspace of the function space $F(X)$ over $\mathbb{F}$
- The set of all symmetric/skew-symmetric matrices form a subspace of the space $M_n(\mathbb{F})$ over $\mathbb{F}$.

NOTE

- If $W_1 \& W_2$ are subspace of V, then so are $W_1 \cap W_2$, $W_1 + W_2$.
- If $W_1 \& W_2$ are subspace of V, then $W_1 \cup W_2$ is subspace of V iff one is contained in another.

LINEAR INDEPENDENCE

Let V be a vector space over $\mathbb{F}$, then a linear combination of vectors $v_1, v_2, \dots, v_n$ in V is a vector

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ for some scalars $c_1, c_2, \dots, c_n$ in $\mathbb{F}$.

The vectors $v_1, v_2, \dots, v_n$ are said to be linearly independent

if $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \Rightarrow c_i = 0$, for every i, otherwise the vectors are said to be linearly dependent.

NOTE

- The set $\{0\}$ is linearly dependent
- The set $\{u\}, u \neq 0$, is linearly independent.
- The set $\{v_1, v_2, \dots, v_n\}$ is linearly dependent if any of v_i is equal to 0.
- The set $\{v_1, v_2, \dots, v_n\}$ is linearly dependent if $v_i = cv_j$ for some $i \neq j, c \in \mathbb{F}$.
- Any subset of a linearly independent set is linearly independent.
- Any superset of a linearly dependent set is linearly dependent
- The empty set ϕ is linearly independent.

SPANNING SET

Let V be a vector space over $\mathbb{F}$, then the subset $S = \{v_1, v_2, \dots, v_n\}$ of V if for any $v \in V$,

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ for some scalars $c_1, c_2, \dots, c_n$ in $\mathbb{F}$.

LINEAR SPAN OF A SET

Let V be a vector space over $\mathbb{F}$, $S = \{v_1, v_2, \dots, v_n\}$ be a subset of V, then the Linear span $L(S)$ of S is given by $L(S) = \{v \in V : v = c_1v_1 + c_2v_2 + \cdots + c_nv_n, c_i \in \mathbb{F}\}$

NOTE

For any subset $S \neq \phi$ of a vector space V over $\mathbb{F}$, $L(S)$ is a subspace of V.

BASIS OF A VECTOR SPACE

A subset S of a vector space V over $\mathbb{F}$, is a basis for V if it is linearly independent and spans V.

EXAMPLE

Then vectors $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ constitute a basis for the vector space $\mathbb{R}^3$.

DIMENSION OF A VECTOR SPACE

A vector space V is finite-dimensional if it has a finite basis its dimension is the no. of elements in the basis. If V does not have a finite basis it is infinite dimensional.

NOTE

- Any subset of a vector space V (of dimension n) having more than n vectors is linearly independent.
- Let S be a subset of a vector space V (of dimension n) having n vectors, is linearly independent iff $L(S) = V$
- Let W be a subspace of a finite dimensional space V, then $\dim W \leq \dim V$
- Let V be a vector space over $\mathbb{F}$ with $\dim V = n$ and $\{v_1, v_2, \ldots, v_n\}$ be a basis for V, if $A \in GL_n(\mathbb{F})$, then the set $\{Av_1, Av_2, \ldots, Av_n\}$ also constitute a basis for V.
- Let V be a vector space over $\mathbb{F}$ with $\dim V = n$ and $\mathbb{F}$ is a finite field, then V is a finite vector space.
- Let $A \in M_n(\mathbb{F})$ and let $W = L\{I, A, A^2, \ldots, A^n, \ldots\}$, then $\dim W \leq n$.
- Let V be a V.S over $\mathbb{F}$ has dimension n and $|\mathbb{F}| = p^k$, for some prime p, then $|V| = p^{nk}$

MATRICES

An $m \times n$ matrix is a rectangular array of mn numbers (real or complex) arranged in an ordered set of m horizontal lines called rows and n vertical lines called columns enclosed in parentheses.

An $m \times n$ matrix usually written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & . & . & a_{2n} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & . & . & . & . & a_{mn} \end{bmatrix}_{m\times n}$$

A matrix $[a_{ij}]_{m\times n}$ is called

- A rectangular matrix if $m \neq n$.
- A square matrix if $m = n$.
- A row matrix if $m = 1$.
- A column matrix if $n = 1$.
- A null matrix if $a_{ij} = 0 \; \forall \; i, j$.

A square matrix A is called

- Diagonal matrix if $a_{ij} = 0 \; \forall \; i \neq j$.
- Scalar matrix if $a_{ij} = 0 \; \forall \; i \neq j \; and \; all \; diagonal \; elements \; a_{ii} \; are \; equal.$
- Unit/identity matrix if if $a_{ij} = 0 \; \forall \; i \neq j \; and \;\; a_{ii} = 1 \; \forall i.$
- Upper (lower) triangular matrix if $a_{ij} = 0 \; \forall \; j < i \; (j > i)$

MATRIX ADDDITION

Two matrices can be added only if they are of same order.

A + B = C. i.e, $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [c_{ij}]_{m\times n}$.

- Matrix under addition is a commutative group denoted by $M_{m\times n}(K)$.

SCALAR MULTIPLICATION

A = $[a_{ij}]_{m\times n}$ and k is a scalar then

kA = $[ka_{ij}]_{m\times n}$.

- $A\cdot 0 = 0\ and\ A\cdot 1 = 1$
- $(k_1 + k_2)A = k_1A + k_2A$
- $(k_1\cdot k_2)A = (k_2\cdot k_1)A$
- $k(A + B) = kA + kB$

MATRIX MULTIPLICATION

A of order $m\times n$ and B of order $p\times q$ then to multiply n should be equal to p.

i.e, $[a_{ij}]_{m\times n} + [b_{ij}]_{n\times q} = [c_{ij}]_{m\times p}$

- multiplication of two matrices should not be commutative. AB≠BA.
- Associative. $(A\cdot B)C = A(B\cdot C)$
- Identity exists only for square matrices.
- Inverse doesn't exist for all matrices.

TRACE

trace is the sum of all diagonal elements of a square matrix and is denoted by trA.

PROPERTIES

- $trA = \sum_{i=1}^{n} a_{ij}$
- $tr(A + B) = trA + trB$
- $tr(kA) = k(trA)$
- $trace\ of\ identity\ matrix\ is\ n.$
- trace of zero matrix is zero.

DETERMINANT

Every square matrix $A = [a_{ij}]_{m\times n}$ is associated with a number called determinant of A and is denoted by |A|, or det A.

Also, It is a continuous map from $M_n(\mathbb{R})$ to $\mathbb{R}$ satisfying

- $\det A^t = \det A$
- Interchanging of any two Rows/ Columns in the determinant will change the sign
- If any 2 rows/columns of A is identical or proportional, then $\det A = 0$
- $\det(kA) = k^n \det A$
- If every element of a raw/column can be expressed as sum of 2 or more terms, then the determinant can be expressed as the sum of 2 or more determinants

PROPERTIES

Let $A, B \in M_n(\mathbb{R})$, then

- $\det A\, B = \det A \det B$
- $\det A^2 = (\det A)^2$
- $\det A^k = (\det A)^k,\ k \in \mathbb{N}$
- $AB = 0 \Rightarrow \det A = 0$ or $\det B = 0$
- B is said to be the Inverse of A, if $AB = BA = I$, in that case A is invertible and $B = A^{-1}$

ADJOINT OF A MATRIX

If $A = [a_{ij}]_{m \times n}$ is a square matrix and c_{ij} is the cofactor of a_{ij} in A. then the transpose of the matrix obtained from A after replacing each element by corresponding cofactor is called the adjoint of A and is denoted by adj A. Thus, Adj A= $[c_{ij}]$'

NOTES

Let $A, B \in M_n(\mathbb{R})$, then

- $adj(AB) = adj(B).adj(A)$
- $A.adj(A) = \det A . I$
- $adj(adj\ A) = (\det A)^{n-2} A$

SOME SPECIAL MATRICES

SYMMETRIC MATRICES

$A \in M_n(\mathbb{R})$ is said to be symmetric if $A^t = A$

- If A, B are symmetric, then the following also , $A \pm B, kA, AB + BA, A^k, AA^t, A^t A, ABA, BAB$
- If A, B are symmetric, then AB is symmetric iff $AB = BA$
- $A \in M_n(\mathbb{R}) \Rightarrow A + A^t$ is symmetric

SKEW-SYMMETRIC MATRICES

$A \in M_n(\mathbb{R})$ is said to be skew symmetric if $A^t = -A$

- If A, B are skew-symmetric, then A^k is skew symmetric if k is odd and is symmetric if k is even
- $A \in M_n(\mathbb{R}) \Longrightarrow A - A^t$ is skew-symmetric
- If A is skew-symmetric the $I - A$ is invertible
- Diagonal elements of a skew-symmetric matrix are zero
- det of skew-symmetric matrices of odd order is zero. ($A = -A' \Rightarrow \det(A) = (-1)^n \det A$)
- det of Skew-symmetric matrices of even order with $\mathbb{Z}$ entries is a perfect square

- det of Skew-symmetric matrices of even order is non-negative
- $A \in M_n(\mathbb{R}) \Rightarrow A = \frac{A+A^t}{2} + \frac{A-A^t}{2}$. (Any square matrix can be represented as sum of symmetric and skew symmetric matrices).
- The vector space $M_n(\mathbb{R})$ can be expressed as the direct sum of set of all symmetric matrices and skew-symmetric matrices.

HERMITIAN MATRICES

$A \in M_n(\mathbb{C})$ is hermitian if $A^* = A$ (self adjoint), where $A^* = (\bar{A})^t$ is the conjugate transpose of A.

- Real symmetric matrices are hermitian, unlike in the case of $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.
- Diagonal entries of a hermitian matrix are real.
- A is hermitian $\Rightarrow trace(A), \det A \in \mathbb{R}$.
- The result holding for real symmetric matrices are true for hermitian matrices also.

SKEW-HERMITIAN MATRICES

$A \in M_n(\mathbb{C})$ is skew-hermitian if $A^* = -A$

- Real skew-symmetric matrices are skew-hermitian.
- Diagonal entries of skew-hermitian matrix must be zero or purely imaginary.
- The results holding for real skew-symmetric matrices are true for skew-hermitian matrices also.
- A is hermitian $\Leftrightarrow iA$ is skew-hermitian.

Cartesian decomposition:

let $A \in M_n(\mathbb{C})$, then A can be expressed as $A = B + iC$, where A is hermitian and B is skew-hermitian $\left(B = \frac{A+A^*}{2}, C = \frac{A-A^*}{2i}\right)$.

ORTHOGONAL MATRIX

$A \in M_n(\mathbb{R})$ is said to be orthogonal if, $AA^t = A^tA = I$.

- A is orthogonal $\Rightarrow \det A = \pm 1$
- Any 2 row/column vectors of an orthogonal matrix form an orthogonal basis for $\mathbb{R}^n$.
- If A, B are orthogonal, then A^{-1}, AB are orthogonal
- $OL_n(\mathbb{R})$: Multiplicative group of all orthogonal matrices of order n.
- Sum or difference of 2 orthogonal matrices need not be orthogonal.
- **Ex:** (general form of 2×2 orthogonal matrix)

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, 0 < \theta \leq 2\pi \Rightarrow A^k = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}, k \in \mathbb{Z}$$

- Any orthogonal matrix of order n is a block representation of 2×2 orthogonal matrices and $[1]$.

UNITARY MATRIX

$A \in M_n(\mathbb{C})$ is said to be unitary if, $AA^* = A^*A = I$.

- Real orthogonal matrices are unitary
- $|\det A| = 1$ (since, $\det A^* = \det A$)
- The result holding for real orthogonal matrices are true for unitary matrices also.

NORMAL MATRIX

$A \in M_n(\mathbb{C})$is said to be normal if, $AA^* = A^*A$.

- Real symmetric, skew-symmetric, orthogonal matrices are normal.
- Hermitian and skew-hermitian matrices are normal.

PERMUTATION MATRIX

$A = [a_{ij}]_{n\times n}$ is said to be a permutation matrix if, $\sum_j a_{ij} = \sum_i a_{ij} = 1,\ \ a_{ij} = 0$ or 1

- A, is a permutation matrix $\Rightarrow A^k = I$, for some $k \in \mathbb{N}$ such that $k/n!$, such least integer k is called as the order of the matrix.
- All permutation matrices are orthogonal.
- Permutation matrices are obtained by permuting rows/columns of identity matrix.
- If A, B are permutation matrices, then AB, BA are permutation matrices.
- Let S_n be the group of all permutations of n symbols under composition, then corr. To any $\sigma \in S_n$, there is a permutation matrix A_σ.
- $A_\sigma^{-1} = A_{\sigma^{-1}}$
- $A_\sigma^{O(\sigma)} = I$

IDEMPOTENT MATRIX

$A \in M_n(\mathbb{R})$ is said to be idempotent if $A^2 = A$, also called projections.

- A non-identity idempotent matrix is singular.
- A is idempotent $\Rightarrow A^t, A^k, I - A$ are idempotent
- If A, B are idempotent, then $A + B$ is idempotent $\Leftrightarrow AB + BA = 0$.

PERIODIC MATRIX

$A \in M_n(\mathbb{R})$ is said to be periodic if $A^k = A$, for some $k(> 1) \in \mathbb{N}$. The least such k is called the period of such matrix.

- Idempotent matrices other than identity matrix are singular.

NILPOTENT MATRIX

ENTRI

$A \in M_n(\mathbb{R})$ is said to be nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.

- Index of nilpotency : the least no. $k \in \mathbb{Z}^+$ such that $A^k = 0, \ (k \le n)$.
- A is nilpotent $\Rightarrow \det A = 0$

INVOLUTARY MATRIX

$A \in M_n(\mathbb{R})$ is said to be involuntary if $A^2 = I$.

- A is involuntary $\Rightarrow \det A = \pm 1$
- A is involuntary $\Rightarrow A^{-1} = A$
- A is involuntary $\Rightarrow \frac{A+I}{2}$ is idempotent

NOTE

- $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \Rightarrow \det A = 3abc - a^3 - b^3 - c^3$.
- $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \Rightarrow \det A = abc + 2fgh - af^2 - bg^2 - ch^2$
- $A, B \in M_n(\mathbb{R})$, then $trace(AB) = trace(BA)$ and $\det(AB) = \det(BA)$.

$M_{m \times n}(\mathbb{R})$ denotes the set of all $m \times n$ matrices with real entries.

RANK OF A MATRIX

Rank of a matrix $A \in M_{m \times n}(\mathbb{R})$ can be defined as,

1. The no. of lin. Independent rows/columns of A.
2. Order of largest non-singular sub matrix of A.
3. Dimension of row/column space of A.
4. The number of non-zero rows in the row-reduced echelon form of A.
5. The order of identity submatrix in the normal form of A.
6. The rank of the lin. Transformation from $\mathbb{R}^n \to \mathbb{R}^m$ corresponding to A.

Usually rank of $A \in M_{m \times n}(\mathbb{R})$ is denoted as $\rho(A)$.

NOTE

let $A, B \in M_n(\mathbb{R})$, then

- $\rho(A) = 0 \iff A = 0$
- $\rho(A) \le \min\{m, n\}$
- $\rho(A + B) \le \rho(A) + \rho(B)$
- $\rho(AB) \le \min\{\rho(A), \rho(B)\}$
- $A \in M_{m \times n}(\mathbb{R}), \rho(AA^t) = \rho(A) = \rho(A^t A)$
- $A, B \in M_n(\mathbb{R}), \rho(A) + \rho(B) - n \le \rho(AB)$

- $A, B \in M_n(\mathbb{R}), \rho(I - AB) = \rho(I - BA)$
- Rank of nonzero skew symmetric matrix is ≥ 2
- Rank of nonzero skew symmetric matrix of odd order n is atmost $n - 1$
- $A \in M_n(\mathbb{R}), \rho(A) \geq \rho(A^2) \geq \rho(A^3) \geq \cdots$
- $A \in M_n(\mathbb{R})$ and $A^2 = A \Rightarrow \rho(A) = trace(A)$ and $\rho(A) + \rho(I - A) = n$.
- No. of lin. Indpt Rows= No. of lin. Indpt Columns.
- $A \in M_{m\times n}(\mathbb{R})$ with $\rho(A) = m \leq n, B \in M_{p\times m}(\mathbb{R}) \Rightarrow \rho(BA) = \rho(B)$.

NULLITY

Nullity of $A \in M_{m\times n}(\mathbb{R}), \eta(A)$ is defined as the dimension of null space of A, i.e dim $N(A)$, where $N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$.

- $\eta(A) = n - \rho(A)$.

IMAGE SPACE (RANGE) OF A

$R(A) = \{Ax \in \mathbb{R}^n : x \in \mathbb{R}^n\}$

- $\rho(A) = \dim\big(R(A)\big)$.
- Column space of $A = R(A)$.
- Column nullity of $A = n - \rho(A) = \eta(A)$.
- Row space of $A = \{y^t A \in \mathbb{R}^n ; y \in \mathbb{R}^m\}$
- Row nullity of $A = m - \rho(A)$.
- $A \in M_n(\mathbb{R}), \eta(A) \leq \eta(A^2) \leq \eta(A^3) \leq \cdots$

Let $A \in M_{m\times n}(\mathbb{R}), X \in \mathbb{R}^n$ and $B \in \mathbb{R}^m$, then the equation $AX = B$ represents m linear equations in n unknowns.

SYSTEM OF LINEAR EQUATIONS

Consider a system of simultaneous linear equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

These equations can be written in a matrix form AX = B.

Where, matrix A is called coefficient matrix. X is called the unknown vector which will be the solution. A system of linear equations may have a unique solution or infinitely many solutions or no solution. If the system has solutions then the system is called consistent. Otherwise, inconsistent.

SOLUTIONS OF SYSTEM OF LINEAR EQUATION

NON-HOMOGENIOUS SYSTEM

$AX = B, B \neq 0$

- The system is said to be consistent if $\rho(A) = \rho(A:B)$
- The system is said to be inconsistent if, $\rho(A) \neq \rho(A:B) \Leftrightarrow \rho(A:B) > \rho(A)$.
- The system has a unique solution $\Leftrightarrow \rho(A) = \rho(A:B) = n$.
- The system has infinite no. of solutions $\Leftrightarrow \rho(A) = \rho(A:B) < n$.
- $m < n \Rightarrow$ system cannot have a unique solution.
  (2 possibilities): 1. $\rho(A) = m = \rho(A:B)$
  2. $\rho(A) < m$
- The number of free variables in $X_{n\times 1}$ is $n - \rho(A)$

Particular case: $m = n$

- The system has a unique solution $\Leftrightarrow \det A \neq 0$ and is given by $X = A^{-1}B$.
- The system has finite no. of solutions $\Leftrightarrow \det A = 0$ and $adj\ (A)B = 0$.
- The system has no solution $\Leftrightarrow \det A = 0$ and $adj\ (A)B \neq 0$.

HOMOGENIOUS SYSTEM

$AX = 0$.

- $\rho(A) = \rho(A:0)$, The system always consistent.
- If $\rho(A) = n$, then the system has unique solution (trivial).
- $\rho(A) < n$, System has infinite no. of solutions.
- $m < n \ \Leftrightarrow \rho(A) < n \ \Leftrightarrow$ System has infinite no. of solutions.
- The solution space of the system is the space $N(A)$

Particular case $m = n$

- System has unique solution $\Leftrightarrow \det A \neq 0$ and is given by $X = 0$
- System has infinite no. of solutions $\Leftrightarrow \det A = 0$.
- System has atleast
- one solution.

EIGEN VALUES AND EIGEN VECTORS

Let $A \in M_n(\mathbb{F})$ and $\Delta_A(x)$ be its char. Polynomial, then the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of $\Delta_A(x)$ is called the eigen values of A.

- **Eigen vector:**

If $\exists v(\neq 0) \in \mathbb{F}^n$ such that $Av = xv$, then x is an eigen value of A and v is an eigen vector corr. to the eigen value x.

- **Algebraic Multiplicity (a.m)**
  a.m of the eigen value x of A is the multiplicity of x, as a root in the equation $\det(xI - A) = 0$
- **Eigen Space**
  Corresponding to the eigen value x of A, $E_x = \{v : (A - xI)v = 0\}$. i.e, E_x is the solution space of the system $(A - xI)v = 0$.
- **Geometric multiplicity (g.m)**
  g.m of $x = \dim E_x$.

NOTE

let $A \in M_n(\mathbb{F})$,

- For any eigen value x of A, $1 \leq g.m(x) \leq a.m(x)$
- If v_1, v_2 are eigen vectors of A corr. to the eigen value x then $c_1v_1 + c_2v_2,\ c_1, c_2 \in \mathbb{F}$ (providing $c_1v_1 + c_2v_2 \neq 0$) is again an eigen vector of A corr. to x.
- v is an eigen vector of A corr. to x $\Leftrightarrow kv$ is an eigen vector of A corr. to x, for any scalar k.
- $g.m(x) = n - \rho(A - xI)$ = no. of lin. Indpt. Eigen vectors corr. to x.
- $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigen values of $A \in M_n(\mathbb{F})$ and $v_1, v_2, \dots, v_k$ are corr. eigen vectors, then the set $\{v_1, v_2, \dots, v_k\}$ is linearly independent.
- If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of $A \in M_3(\mathbb{F})$,
  $\Delta_A(x) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1.\lambda_2 + \lambda_2\lambda_3 + \lambda_1.\lambda_3)x - \lambda_1.\lambda_2.\lambda_3$
- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of $A \in M_n(\mathbb{F})$, need not be distinct, then
  $trace(A) = \sum_{i=1}^{n} \lambda_i$ & $\det A = \prod_{i=1}^{n} \lambda_i$
- Eigen values of triangular matrices are exactly the diagonal entries.

CAYLEY HAMILTON THEOREM

Every square matrix satisfies its characteristic equation, i.e, $\Delta_A(x) = 0$

LINEAR TRANSFORMATION

Let V and W be two vector spaces over same field F. Then the map $T: V \to W$ is said to be a linear transformation if

1. $T(v_1 + v_2) = T(v_1) + T(v_2) \forall v_1, v_2 \in V$
2. $T(\alpha v) = \alpha T(v) \forall \alpha \in F and v \in V$

EXAMPLES

- $R^n = \{(a_1, a_2, \dots, a_n\}, a_i \in R$ be vector space over field R, then
  - $T: R^n \to R^n$ such that $T(a_1, a_2, \dots, a_n) = (a_n, a_{n-1}, \dots, a_1)$
  - $T: R^n \to R^n$ such that $T(a_1, a_2, \dots, a_n) = (0, a_1, \dots, a_{n-1})$
  - $T: R^n \to R^n$ such that $T(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_{n-1}, 0)$

- $R[x] = \{P(x)|P(x) is\ polynomial\ over\ field\ R\}$, then
  - $T: R[x] \to R[x]$ such that $T\big(P(x)\big) = P'(x)$
  - $T: R[x] \to R$ suich that $T\big(P(x)\big) = P(0)$

- Identify transformation $I(x) = x$ is linear transformation from $V\ onto\ V$.

- Zero transformation $0(x) = 0$ is trivial transformation.

LINEAR MAP

Let V and W be two V.S over $\mathbb{F}$, a map $T : V \to W$ is said to be a linear map if $T(cv_1 + v_2) = cTV_1 + Tv_2,\ v_1, v_2 \in V$ and $c \in \mathbb{F}$.

KERNEL OF A LINEAR TRANSFORMATION

$N(T) = Ker(T) = \{v \in V : Tv = 0\}$

- $Ker(T)$ is a subspace of V and $\dim Ker(T)$is the nullity $n(T)$ of T.
- Let $T: V \to V$ be a linear transformation, then $Ker(T) \subseteq Ker(T^2) \subseteq \cdots$ and $n(T) \leq n(T^2) \leq \cdots$

RANGE SPACE

Let $T: V \to W$ be a linear transformation, then $Im(T) = R(T) = \{Tv : v \in V\}$

- $R(T)$ is a subspace of W and $\dim R(T)$ is the rank $\rho(T)$ of T

RANK-NULLITY THEOREM

$$\rho(T) + n(T) = \dim V$$

- Let $T: V(\mathbb{F}) \to W(\mathbb{F})$ be linear, then $\{v_1, v_2, \dots, v_n\}$ spans $V \Rightarrow \{Tv_1, \dots, Tv_n\}$ spans $R(T)$
- Let $\dim V = \dim W = n$, and Let $B = \{v_1, v_2, \dots, v_n\}$ is basis for V and $B' = \{w_1, w_2, \dots, w_n\}$ is for W, then $\exists\ T: V \to W$ such that $Tv_1 = w_1, Tv_2 = w_2, \dots, Tv_n = w_n$.
- $T: V \to W$, non singular (Injective) $\Leftrightarrow Ker(T) = \{0\}$, here $\dim V = \dim R(T)$
- $T: V \to W$, $\dim V = \dim W, T$ is injective $\Leftrightarrow$ surjective.

KINDS OF LINEAR TRANSFORMATION

LINEAR OPERATOR

Let $V(F)$ be vector space then any linear transformation from V into V is called linear operator on V.

SINGULAR

A linear transformation called singular if $\exists$ a non zero vector x such that $T(x) = 0$, that is linear transformation is singular if $\ker T \neq 0$ that is $\eta(T) \geq 1$

ENTRI

NON-SINGULAR

A linear transformation is called non-singular if $T(x) = 0 \Rightarrow x = 0$ that is linear transformation is non-singular if $\ker T = \{0\}$.

- $T: V \to W$, non singular (Injective) $\Leftrightarrow Ker(T) = \{0\}$, here $\dim V = \dim R(T)$
- $T: V \to W$, $\dim V = \dim W, T$ is injective $\Leftrightarrow$ surjective.

INVERTIBLE LINEAR TRANSFORMATION

Let $V(F)$ and $V'(F)$ be vector space over F and let $T: V \to V'$ be a linear transformation such that T is one-one and onto then T is called invertible. If there exist a linear transformation $S: V' \to V$ such that $TS = 1 = ST$.

COROLLARY

Let V, W and Z be vector space over F. Let T be linear transformation from V to W and S is linear transformation from W $into$ Z. Then SoT, composite function is defined by

$(SoT)(x) = S\big(T(x)\big), \forall x \in V$ is a linear transformation from V to Z.

- Let V and W are two finite dimensional vector space such that $\dim V = \dim W$ and $T: V \to W$ a linear transformation. Then the following are equivalent.
  - ➢ T is invertible
  - ➢ T is non-singular
  - ➢ T is onto
  - ➢ If $\{v_1, v_2, \dots, v_n\}$ is a basis of V, then $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis of W.
- Let T be a linear transformation from V into W, then T is non-singular iff T carries each linearly independent subset of V onto a linearly independent subset of W.
- Let $T: V \to W$ is linear transformation such that $\dim V = \dim W$.
  Then T is invertible $\Leftrightarrow$ T is invertible $\Leftrightarrow$ T is $1-1$ $\Leftrightarrow$ T is non-singular $\Leftrightarrow$ T is onto.
- If T and U are linear transformation on a finite dimensional vector space V such that $TU = I$ then U are invertible and $T^{-1} = U$.

LINEAR FUNCTIONAL

Let $V(F)$ be vector space over field F. Then a maping $f: V \to f$ is called linear transformational on V if $f(ax + by) = af(x) + bf(y) \forall x, y \in V$ and $a, b \in F$.

- $f(0) = 0, 0 \in V$ and $0 \in F$
- $f(-x) = -f(x) \forall x \in V$

PROPERTIES OF LINEAR TRANSFORMATION

Let $T, V \to V'$ is a linear transformation from $V(F)$ to $V'(F)$

- If T is linear, then $T(0) = 0$
- T is linear iff $T(cx + y) = cT(x) + T(y) \forall x, y \in V, c \in F$
- If T is linear $T(x - y) = T(x) - T(y) \forall x, y \in V$

- T is linear iff $T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)\; a_i \in F, x_i \in V$
- Let V and W be vector spaces and let $T, V \to W$ be linear. If $\beta = \{V_1, V_2, \dots, V_n\}$ is a basis for V. Then $R(T) = span(T(\beta)) = span(\{T(V_1), T(V_2), \dots, T(V_n)\})$.
- Let V be vector space, $T: V \to V$ be linear operator. If V is finite dimensional, $nullity\ (T) - rank(T) = dim(V) \Longrightarrow T$ is bijective.
- Let $V\ and\ W$ be vector space and $T: V \to W$ be linear. Then T is one to one iff $N(T) = \{0\}$.
- Let $V\ and\ W$ be vector space over F. If $\{v_1, v_2, \dots, v_n\}$ is a basis of v then for any $w_1, w_2, \dots, w_n\ in\ W$, there exist exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i, i = 1,2, \dots, n$.
- Let $V\ and\ W$ are vector space and $T: V \to W$ be linear transformation then
    - If $x_1, x_2, \dots, x_n$ are linearly independent in V, then $T(x_1), T(x_2), \dots, T(x_n)$ may or may not be linearly independent on W.
    - If $x_1, x_2, \dots, x_k$ are linearly dependent in V, then $T(x_1), T(x_2), \dots T(x_k)$ are linearly dependent in W.
    - If $T(x_1), T(x_2), \dots, T(x_k)$ are linearly independent in W then $x_1, x_2, \dots, x_k$ are linearly independent in V.
    - If $T(x_1), T(x_2), \dots, T(x_k)$ are linearly dependent in W, then $x_1, x_2, \dots, x_k$ may or may not linearly dependent on V.
- Let $V\ and\ W$ are vector spaces and $T: V \to W$ be one-one linear transformation, then
    - If $x_1, x_2, \dots, x_k$ are linearly independent vectors in V iff $T(x_1), T(x_2), \dots, T(x_k)$ are linearly independent in W.
    - $x_1, x_2, \dots, x_k$ are linearly dependent vectors in V iff $T(x_1), T(x_2), \dots, T(x_k)$ are linearly dependent in W.
    - $B = \{v_1, v_2, \dots, v_n\}$ is a basis for $V\ and\ T$ is one-one and onto, then $T(B) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W.
- Let $V\ and\ W$ be finite dimensional and $T: V \to W$ be linear
    - If $\dim(V) < \dim(W) \Rightarrow T$ is not onto
    - If $\dim(V) > \dim(W) \Rightarrow T$ is not one-one
- T be on invertible linear transformation from $V \to W$ then V is finite dimensional iff W is finite dimensional and $\dim V = \dim W$.
- Let $V\ and\ W$ be vector spaces over F. Then set of all linear transformations from $V \to W$ denoted by $L(V, W)$. If $V = W$ then we write $L(V)$.
- Let $V = R[x] for\ j \geq 1, T_j(p(x)) = p^{(j)}(x), jth$ derivative of $p(x)$ then $\{T_1, T_2, \dots, T_n\}$ is linearly independent subset of $L(V)$.

MATRIX REPRESENTATION OF LINEAR TRANSFORMATION

Let V and W be two vector spaces with $\dim V = n$ and $\dim W = m$, let $B = \{v_1, v_2, \dots, v_n\}$ and $B' = \{w_1, w_2, \dots, w_m\}$ be respective bases.

$$Tv_1 = c_{11}w_1 + c_{21}w_2 + \dots + c_{m1}w_m$$

$$Tv_2 = c_{12}w_1 + c_{22}w_2 + \dots + c_{m2}w_m$$

.

.

$$Tv_n = c_{1n}w_1 + c_{2n}w_2 + \dots + c_{mn}w_m$$

$$[T]_{B'}^{B} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ . & . & . & . & . \\ . & . & . & & . \\ . & . & . & . & . \\ c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mn} \end{pmatrix}$$

Column vectors are coordinate vectors of Tv_i.

DIAGONALIZATION

SIMILAR MATRICES

Let $A, B \in M_n(\mathbb{R})$, then A is similar to B, denoted by $A \sim B$, if there exist an invertible P such that $P^{-1}AP = B$

- $A \sim B \Rightarrow \rho(A) = \rho(B)$
- $A \sim B \Rightarrow C_A(x) = C_B(x)$
- $A \sim B \Rightarrow \det A = \det B$
- $A \sim B \Rightarrow m_A(x) = m_B(x)$
- $A \sim B \Rightarrow Trace(A) = Trace(B)$
- $A \sim B^{-1}AB$, for any invertible B
- $A, B \in M_n(\mathbb{F}) \Rightarrow$ eigen values of AB and BA are equal $\Rightarrow Trace(BA) = Trace(AB)$ and $\det BA = \det AB$
- $A \sim B \Rightarrow A = PBP^{-1}$, Suppose x is an eigen vector of A corresponding to the eigen value λ, then $P^{-1}x$ is an eigen vector of B corresponding value λ.
- A is congruent to B, if there exist an invertible P such that $P^t AP = B$
- $A, B \in M_n(\mathbb{R})$ with any one of A or B, invertible
  $\Rightarrow AB = BA$
  $\Longrightarrow C_{BA}(x) = C_{AB}(x)$
  $\Rightarrow m_{BA}(x) = m_{AB}(x)$
  $\Rightarrow \rho(BA) = \rho(AB)$
- Let $A, B \in M_n(\mathbb{R})$ and both are not invertible, then $\rho(AB)$ need not be equal to $\rho(BA)$

DIAGONALIZABILITY

$A \in M_n(\mathbb{F})$ is said to be diagonalizable if $\exists P$ such that $P^{-1}AP = D$, D is a diagonal matrix

NOTE

- A is diagonalizable $\Leftrightarrow m_A(x)$ is a product of distinct linear factors.
- $a.m(x) = g.m(x)$, for all eigen values x of $A \Leftrightarrow A$ is Diagonalizable
- $\exists\ n$ linearly independent eigen vectors for $A \Leftrightarrow A$ is diagonalizable
- If all eigen values of A are distinct $\Rightarrow A$ is diagonalizable
- Zero matrix is diagonalizable over any field.
- Diagonal matrices are diagonalizable over any field
- Scalar matrices are diagonalizable over any field
- Idempotent matrices are diagonalizable over any field
- Involuntary matrices are diagonalizable over any field F with $char.F \neq 2$

- Non-zero nilpotent matrices are not diagonalizable over any field
- Real symmetric matrices are diagonalizable over $\mathbb{R}$
- Hermitian matrices are diagonalizable over $\mathbb{C}$
- Rank 1 matrices are diagonalizable over any field
- $A^3 = A \Rightarrow A$ is diagonalizable over R
- $A^3 = I \Rightarrow A$ is diagonalizable over C
- $A^3 = -A \Rightarrow A$ is diagonalizable over C
- $A^4 = I$ but $A^2 \neq I \Rightarrow A$ is diagonalizable only over $\mathbb{C}$
- Orthogonal matrices are diagonalizable over $\mathbb{C}$
- Normal matrices are diagonalizable over $\mathbb{C}$

ORDERED BASIS

Ordered basis of a vector space is a basis with specific order. (elements will be in a specific order)

EXAMPLE

B={(1, 0), (0,1)} and B'={(0, 1), (1, 0)} are two different bases of $\mathbb{R}^2$

CHANGE OF BASIS

Let $\{e_1, e_2, \ldots, e_n\}$ and $\{e_1', e_2', \ldots, e_n'\}$ be two arbitrary bases of a vector space V over a field F, we can suppose that $e_i' = \sum_{j=1}^{n} a_{ij}\, e_j$, where $a_{ij} \in F$.

Putting i = 1, 2, 3,…,n we get the coefficient matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & . & . \\ . & . & . & & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Then, the transpose of the above matrix of coefficients is called the transition matrix P from the old basis $\{e_i\}$ to $\{e_j'\}$ and that

P =

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & . & . \\ . & . & . & & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

$\therefore$ $\{e_1', e_2', \ldots, e_n'\}$ is linearly independent set and hence the inverse of the matrix P exists.

Of course, $P^{-1}$is the transition matrix from the basis $\{e_1', e_2', \ldots, e_n'\}$ to the basis $\{e_1, e_2, \ldots, e_n\}$.

CANONICAL FORMS

JORDAN CANONICAL FORM

JORDAN BLOCK: An n×n matrix with λ on the diagonal, 1 on the super diagonal and all other entries zero, is called a jordan block of order n with eigen value λ and is denoted by $J_n(\lambda)$.

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

JORDAN CANONICAL FORM: A matrix M with square matrices on the diagonal and other entries zero is called block diagonal matrix and it is called jordan canonical form if the matrices on diagonal are Jordan blocks.

i.e, $M = \begin{pmatrix} A & O & O \\ O & B & O \\ O & O & C \end{pmatrix}$ is called block diagonalmatrix where, A, B and C are square matrices. This M is called Jordan canonical form if all A, B and C are Jordan blocks.

INNER PRODUCT SPACE

Let V be vector space over F. An inner product map $< \cdot, \cdot > : V \times V \to F$ that satisfies following three properties.

1. $< \alpha u_1 + \beta u_2, \ v \ > = \alpha < u_1 + v > + \beta < u_2 + v > \ \ \forall \ \alpha, \beta \ \in F \ and \ u_1, \ u_2 \in \ V$
2. $< \ u, \ v > = \ \langle \overline{\boldsymbol{v}, \boldsymbol{u}} \rangle$
3. $< \ u, u \ > \ \geq 0 \ and \ = 0 \ iff \ u = 0.$

A vector space V together with an inner product on it is called inner product space.

EXAMPLE

- Let V = C[0, 1] (set of all scalar valued continuous functions defined on [0, 1].

  Define inner product $< \ f, g \ > = \int_0^1 f(t) \ \overline{g(t)} \ dt$

  ⇒ V is real inner product space with given inner product. Above defined inner product is standard inner product on V.

ORTHOGONAL SET

A subset X of V is said to be an orthogonal set if for any x, y ∈ X, $< x, y > = 0$ whenever x≠y.

ORTHONORMAL SET

A subset X of V is called an orthonormal set if

1. $\|x\| = 1 \ \forall \ x \in X.$
2. $< x, y > = 0 \ for \ x, y \ \in X \ whenever \ x \neq y.$

NOTE

A basis of an inner product space that consists of mutually orthogonal unit vectors is called an orthonormal basis.

QUADRATIC FORM

A homogenous expression of second degree in any number of variables is called quadratic form.

A real quadratic form $X^T AX$ in n variables is said to be

1. Positive definite if $X^T AX > 0 \; \forall \; X \neq 0$
2. Negative definite if $X^T AX < 0 \; \forall \; X \neq 0$
3. Positive semi definite if $X^T AX \geq 0 \; \forall \; X \neq 0$
4. Negative semi definite if $X^T AX \leq 0 \; \forall \; X \neq 0$
5. Indefinite if $X^T AX$ attains negative and positive values for some X.