# SYSTEM OF LINEAR EQUATIONS

## MATRICES

An $m \times n$ matrix is a rectangular array of mn numbers (real or complex) arranged in an ordered set of m horizontal lines called rows and n vertical lines called columns enclosed in parentheses.

An $m \times n$ matrix usually written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

A matrix $[a_{ij}]_{m\times n}$ is called

- A rectangular matrix if $m \neq n$.
- A square matrix if $m = n$.
- A row matrix if $m = 1$.
- A column matrix if $n = 1$.
- A null matrix if $a_{ij} = 0 \ \forall \ i, j$.

A square matrix A is called

- Diagonal matrix if $a_{ij} = 0 \ \forall \ i \neq j$.
- Scalar matrix if $a_{ij} = 0 \ \forall \ i \neq j$ *and all diagonal elements* $a_{ii}$ *are equal*.
- Unit/identity matrix if if $a_{ij} = 0 \ \forall \ i \neq j$ *and* $a_{ii} = 1 \ \forall i$.
- Upper (lower) triangular matrix if $a_{ij} = 0 \ \forall \ j < i \ (j > i)$

### MATRIX ADDDITION

Two matrices can be added only if they are of same order.

A + B = C. i.e, $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [c_{ij}]_{m\times n}$ .

- Matrix under addition is a commutative group denoted by $M_{m\times n}(K)$.

### SCALAR MULTIPLICATION

A = $[a_{ij}]_{m\times n}$ and k is a scalar then

kA = $[ka_{ij}]_{m\times n}$.

- $A \cdot 0 = 0$ *and* $A \cdot 1 = 1$
- $(k_1 + k_2)A = k_1A + k_2A$
- $(k_1 \cdot k_2)A = (k_2 \cdot k_1)A$
- $k(A + B) = kA + kB$

## MATRIX MULTIPLICATION

A of order $m \times n$ and B of order $p \times q$ then to multiply n should be equal to p.

i.e, $[a_{ij}]_{m\times n} + [b_{ij}]_{n\times q} = [c_{ij}]_{m\times p}$

- multiplication of two matrices should not be commutative. AB≠BA.
- Associative. $(A \cdot B)C = A(B \cdot C)$
- Identity exists only for square matrices.
- Inverse doesn't exist for all matrices.

## TRACE

trace is the sum of all diagonal elements of a square matrix and is denoted by trA.

### PROPERTIES

- $trA = \sum_{i=1}^{n} a_{ij}$
- $tr(A + B) = trA + trB$
- $tr(kA) = k(trA)$
- $trace\ of\ identity\ matrix\ is\ n.$
- trace of zero matrix is zero.

## DETERMINANT

Every square matrix $A = [a_{ij}]_{m\times n}$ is associated with a number called determinant of A and is denoted by |A|, or det A.

Also, It is a continuous map from $M_n(\mathbb{R})$ to $\mathbb{R}$ satisfying

- $\det A^t = \det A$
- Interchanging of any two Rows/ Columns in the determinant will change the sign
- If any 2 rows/columns of $A$ is identical or proportional, then $\det A = 0$
- $\det(kA) = k^n \det A$
- If every element of a raw/column can be expressed as sum of 2 or more terms, then the determinant can be expressed as the sum of 2 or more determinants

### PROPERTIES

Let $A, B \in M_n(\mathbb{R})$, then

- $\det A\,B = \det A \det B$
- $\det A^2 = (\det A)^2$
- $\det A^k = (\det A)^k,\ k \in \mathbb{N}$
- $AB = 0 \Rightarrow \det A = 0$ or $\det B = 0$
- B is said to be the Inverse of $A$, if $AB = BA = I$, in that case A is invertible and $B = A^{-1}$

## ADJOINT OF A MATRIX

If $A = [a_{ij}]_{m\times n}$ is a square matrix and $c_{ij}$ is the cofactor of $a_{ij}$ in A. then the transpose of the matrix obtained from A after replacing each element by corresponding cofactor is called the adjoint of A and is denoted by adj A. Thus, Adj A= $[c_{ij}]'$

ENTRI

## NOTES

Let $A, B \in M_n(\mathbb{R})$, then

- $adj(AB) = adj(B).adj(A)$
- $A.adj(A) = \det A . I$
- $adj(adj\ A) = (\det A)^{n-2} A$

## DETERMINANT OF BLOCK MATRICES

- $P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \Rightarrow \det P = \det A \ \det D$ , ($A, D$ are square matrices)
- $A, B, C, D \in M_n(\mathbb{R})$ are pair wise commuting and $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow \det E = \det(AD - BC)$

## SOME SPECIAL MATRICES

### SYMMETRIC MATRICES

$A \in M_n(\mathbb{R})$ is said to be symmetric if $A^t = A$

- If $A, B$ are symmetric, then the following also , $A \pm B, kA, AB + BA, A^k, AA^t, A^tA, ABA, BAB$
- If $A, B$ are symmetric, then $AB$ is symmetric iff $AB = BA$
- $A \in M_n(\mathbb{R}) \Rightarrow A + A^t$ is symmetric

### SKEW-SYMMETRIC MATRICES

$A \in M_n(\mathbb{R})$ is said to be skew symmetric if $A^t = -A$

- If $A, B$ are skew-symmetric, then $A^k$ is skew symmetric if $k$ is odd and is symmetric if $k$ is even
- $A \in M_n(\mathbb{R}) \Longrightarrow A - A^t$ is skew-symmetric
- If $A$ is skew-symmetric the $I - A$ is invertible
- Diagonal elements of a skew-symmetric matrix are zero
- det of skew-symmetric matrices of odd order is zero. ($A = -A' \Rightarrow \det(A) = (-1)^n \det A$)
- det of Skew-symmetric matrices of even order with $\mathbb{Z}$ entries is a perfect square
- det of Skew-symmetric matrices of even order is non-negative
- $A \in M_n(\mathbb{R}) \Rightarrow A = \frac{A+A^t}{2} + \frac{A-A^t}{2}$. (Any square matrix can be represented as sum of symmetric and skew symmetric matrices).
- The vector space $M_n(\mathbb{R})$ can be expressed as the direct sum of set of all symmetric matrices and skew-symmetric matrices.

### HERMITIAN MATRICES

$A \in M_n(\mathbb{C})$ is hermitian if $A^* = A$ (self adjoint), where $A^* = (\bar{A})^t$ is the conjugate transpose of $A$.

- Real symmetric matrices are hermitian, unlike in the case of $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.
- Diagonal entries of a hermitian matrix are real.
- $A$ is hermitian $\Rightarrow trace(A), \det A \in \mathbb{R}$.

- The result holding for real symmetric matrices are true for hermitian matrices also.

## SKEW-HERMITIAN MATRICES

$A \in M_n(\mathbb{C})$ is skew-hermitian if $A^* = -A$

- Real skew-symmetric matrices are skew-hermitian.
- Diagonal entries of skew-hermitian matrix must be zero or purely imaginary.
- The results holding for real skew-symmetric matrices are true for skew-hermitian matrices also.
- $A$ is hermitian $\Leftrightarrow iA$ is skew-hermitian.

**Cartesian decomposition:**

let $A \in M_n(\mathbb{C})$, then $A$ can be expressed as $A = B + iC$, where $A$ is hermitian and $B$ is skew-hermitian $\left(B = \frac{A+A^*}{2}, C = \frac{A-A^*}{2i}\right)$.

## ORTHOGONAL MATRIX

$A \in M_n(\mathbb{R})$ is said to be orthogonal if, $AA^t = A^tA = I$.

- $A$ is orthogonal $\Rightarrow \det A = \pm 1$
- Any 2 row/column vectors of an orthogonal matrix form an orthogonal basis for $\mathbb{R}^n$.
- If $A, B$ are orthogonal, then $A^{-1}, AB$ are orthogonal
- $OL_n(\mathbb{R})$: Multiplicative group of all orthogonal matrices of order $n$.
- Sum or difference of 2 orthogonal matrices need not be orthogonal.
- **Ex:** (general form of $2 \times 2$ orthogonal matrix)
$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, 0 < \theta \le 2\pi \Rightarrow A^k = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}, k \in \mathbb{Z}$$
- Any orthogonal matrix of order $n$ is a block representation of $2 \times 2$ orthogonal matrices and $[1]$.

## UNITARY MATRIX

$A \in M_n(\mathbb{C})$ is said to be unitary if, $AA^* = A^*A = I$.

- Real orthogonal matrices are unitary
- $|\det A| = 1$ (since, $\det A^* = \det A$)
- The result holding for real orthogonal matrices are true for unitary matrices also.

## NORMAL MATRIX

$A \in M_n(\mathbb{C})$is said to be normal if, $AA^* = A^*A$.

- Real symmetric, skew-symmetric, orthogonal matrices are normal.
- Hermitian and skew-hermitian matrices are normal.

## PERMUTATION MATRIX

$A = [a_{ij}]_{n\times n}$ is said to be a permutation matrix if, $\sum_j a_{ij} = \sum_i a_{ij} = 1, \ \ a_{ij} = 0$ or $1$

- $A$, is a permutation matrix $\Rightarrow A^k = I$, for some $k \in \mathbb{N}$ such that $k/n!$ , such least integer $k$ is called as the order of the matrix.
- All permutation matrices are orthogonal.
- Permutation matrices are obtained by permuting rows/columns of identity matrix.
- If $A, B$ are permutation matrices, then $AB, BA$ are permutation matrices.
- Let $S_n$ be the group of all permutations of $n$ symbols under composition, then corr. To any $\sigma \in S_n$, there is a permutation matrix $A_\sigma$.
- $A_\sigma^{-1} = A_{\sigma^{-1}}$
- $A_\sigma^{O(\sigma)} = I$

## IDEMPOTENT MATRIX

$A \in M_n(\mathbb{R})$ is said to be idempotent if $A^2 = A$, also called projections.

- A non-identity idempotent matrix is singular.
- $A$ is idempotent $\Rightarrow A^t, A^k, I - A$ are idempotent
- If $A, B$ are idempotent, then $A + B$ is idempotent $\Leftrightarrow AB + BA = 0$.

## PERIODIC MATRIX

$A \in M_n(\mathbb{R})$ is said to be periodic if $A^k = A$, for some $k(> 1) \in \mathbb{N}$. The least such $k$ is called the period of such matrix.

- Idempotent matrices other than identity matrix are singular.

## NILPOTENT MATRIX

$A \in M_n(\mathbb{R})$ is said to be nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.

- Index of nilpotency : the least no. $k \in \mathbb{Z}^+$ such that $A^k = 0, \ \ (k \leq n)$.
- $A$ is nilpotent $\Rightarrow \det A = 0$

## INVOLUTARY MATRIX

$A \in M_n(\mathbb{R})$ is said to be involuntary if $A^2 = I$.

- $A$ is involuntary $\Rightarrow \det A = \pm 1$
- $A$ is involuntary $\Rightarrow A^{-1} = A$
- $A$ is involuntary $\Rightarrow \frac{A+I}{2}$ is idempotent

## NOTE

- $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \Rightarrow \det A = 3abc - a^3 - b^3 - c^3.$
- $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \Rightarrow \det A = abc + 2fgh - af^2 - bg^2 - ch^2$
- $A, B \in M_n(\mathbb{R})$, then $trace(AB) = trace(BA)$ and $\det(AB) = \det(BA)$.

$M_{m\times n}(\mathbb{R})$ denotes the set of all $m \times n$ matrices with real entries.

## RANK OF A MATRIX

Rank of a matrix $A \in M_{m\times n}(\mathbb{R})$ can be defined as,

1. The no. of lin. Independent rows/columns of $A$.
2. Order of largest non-singular sub matrix of $A$.
3. Dimension of row/column space of $A$.
4. The number of non-zero rows in the row-reduced echelon form of $A$.
5. The order of identity submatrix in the normal form of $A$.
6. The rank of the lin. Transformation from $\mathbb{R}^n \to \mathbb{R}^m$ corresponding to $A$.

Usually rank of $A \in M_{m\times n}(\mathbb{R})$ is denoted as $\rho(A)$.

### NOTE

let $A, B \in M_n(\mathbb{R})$, then

- $\rho(A) = 0 \Leftrightarrow A = 0$
- $\rho(A) \le \min\{m, n\}$
- $\rho(A + B) \le \rho(A) + \rho(B)$
- $\rho(AB) \le \min\{\rho(A), \rho(B)\}$
- $A \in M_{m\times n}(\mathbb{R}), \rho(AA^t) = \rho(A) = \rho(A^tA)$
- $A, B \in M_n(\mathbb{R}), \rho(A) + \rho(B) - n \le \rho(AB)$
- $A, B \in M_n(\mathbb{R}), \rho(I - AB) = \rho(I - BA)$
- Rank of nonzero skew symmetric matrix is $\ge 2$
- Rank of nonzero skew symmetric matrix of odd order $n$ is atmost $n - 1$
- $A \in M_n(\mathbb{R}), \rho(A) \ge \rho(A^2) \ge \rho(A^3) \ge \cdots$
- $A \in M_n(\mathbb{R})$ and $A^2 = A \Rightarrow \rho(A) = trace(A)$ and $\rho(A) + \rho(I - A) = n$.
- No. of lin. Indpt Rows= No. of lin. Indpt Columns.
- $A \in M_{m\times n}(\mathbb{R})$ with $\rho(A) = m \le n, B \in M_{p\times m}(\mathbb{R}) \Rightarrow \rho(BA) = \rho(B)$.
- $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ & $A, B, C$ and $D$ are of same order $\Rightarrow$
  $\rho(E) = \rho(A) + \rho(D - CA^{-1}B)$, if $A$ is invertible.
  $\rho(E) = \rho(A) + \rho(D)$, if $A$ is not invertible.

### NULLITY

Nullity of $A \in M_{m\times n}(\mathbb{R}), \eta(A)$ is defined as the dimension of null space of $A$, i.e dim $N(A)$, where $N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$.

- $\eta(A) = n - \rho(A)$.

### IMAGE SPACE (RANGE) OF A

$R(A) = \{Ax \in \mathbb{R}^n : x \in \mathbb{R}^n\}$

- $\rho(A) = \dim(R(A))$.
- Column space of $A = R(A)$.

- Column nullity of $A = n - \rho(A) = \eta(A)$.
- Row space of $A = \{y^t A \in \mathbb{R}^n ; y \in \mathbb{R}^m\}$
- Row nullity of $A = m - \rho(A)$.
- $A \in M_n(\mathbb{R}), \eta(A) \leq \eta(A^2) \leq \eta(A^3) \leq \cdots$

Let $A \in M_{m\times n}(\mathbb{R}), X \in \mathbb{R}^n$ and $B \in \mathbb{R}^m$, then the equation $AX = B$ represents $m$ linear equations in $n$ unknowns.

## SYSTEM OF LINEAR EQUATIONS

Consider a system of simultaneous linear equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

These equations can be written in a matrix form AX = B.

Where, matrix A is called coefficient matrix. X is called the unknown vector which will be the solution. A system of linear equations may have a unique solution or infinitely many solutions or no solution. If the system has solutions then the system is called consistent. Otherwise, inconsistent.

## SOLUTIONS OF SYSTEM OF LINEAR EQUATION

### NON-HOMOGENIOUS SYSTEM

$AX = B, B \neq 0$

- The system is said to be consistent if $\rho(A) = \rho(A{:}B)$
- The system is said to be inconsistent if, $\rho(A) \neq \rho(A{:}B) \Leftrightarrow \rho(A{:}B) > \rho(A)$.
- The system has a unique solution $\Leftrightarrow \rho(A) = \rho(A{:}B) = n$.
- The system has infinite no. of solutions $\Leftrightarrow \rho(A) = \rho(A{:}B) < n$.
- $m < n \Rightarrow$ system cannot have a unique solution.
  (2 possibilities): 1. $\rho(A) = m = \rho(A{:}B)$
  2. $\rho(A) < m$
- The number of free variables in $X_{n\times 1}$ is $n - \rho(A)$

**Particular case:** $m = n$

- The system has a unique solution $\Leftrightarrow \det A \neq 0$ and is given by $X = A^{-1}B$.
- The system has finite no. of solutions $\Leftrightarrow \det A = 0$ and $adj\ (A)B = 0$.
- The system has no solution $\Leftrightarrow \det A = 0$ and $adj\ (A)B \neq 0$.

## HOMOGENIOUS SYSTEM

$AX = 0.$

- $\rho(A) = \rho(A{:}\,0)$, The system always consistent.
- If $\rho(A) = n$, then the system has unique solution (trivial).
- $\rho(A) < n$, System has infinite no. of solutions.
- $m < n \iff \rho(A) < n \iff$ System has infinite no. of solutions.
- The solution space of the system is the space $N(A)$

**Particular case** $m = n$

- System has unique solution $\Leftrightarrow \det A \neq 0$ and is given by $X = 0$
- System has infinite no. of solutions $\Leftrightarrow \det A = 0.$
- System has atleast
- one solution.

# VECTOR SPACES

## VECTOR SPACE

A non-empty set $V$ is said to be a vector space over a scalar field $\mathbb{F}$ together with operations, addition and scalar multiplication, if it satisfies the following axioms:

1. If $x, y \in V$, then $x + y \in V$
2. $(x + y) + z = x + (y + z)$ for every $x, y, z \in V$ (associativity)
3. There exist $0 \in V$ such that $x + 0 = x$ for every $x \in V$ (existence of additive identity)
4. For every $x \in V$ there exist $-x \in V$ such that $x + (-x) = 0$ (existence of additive inverses)
5. $x + y = y + x$ for every $x, y \in V$ (commutativity)
6. If $c \in \mathbb{F}$ and $x \in V$, then $cx \in V$
7. $c(x + y) = cx + cy$ for every $c \in \mathbb{F}$ and every $x, y \in V$
8. $(c + d)x = cx + dx$ for every $c, d \in \mathbb{F}$ and every $x \in V$
9. $(cd)x = c(dx)$ for every $c, d \in \mathbb{F}$ and every $x \in V$ ;
10. $1x = x$ for every $x \in V$

## EXAMPLES

- Zero space or trivial space $V = \{0\}$ over any field
- $V = F$ over any field $\mathbb{F}$
- $V = \mathbb{F}$ over any sub-field of $\mathbb{F}$
- $V = \mathbb{F}^n$ over $\mathbb{F}$ $(\mathbb{Z}_p^n$ over $\mathbb{Z}_p)$
- Polynomial Space $P(X)$ over $\mathbb{F}$, $P(X) = \{p(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots \ | a_i \in \mathbb{F}\}$
- Polynomial Space $P_n(X)$ over $\mathbb{F}$,
  $P_n(X) = \{a_0 + a_1x + a_2x^2 + \cdots + a_mx^m \mid a_i \in \mathbb{F} \,\&\, 0 \leq m \leq n\}$

- Function space $F(X)$ over $\mathbb{F}$, $(X \neq \phi)$, $F(X) = \{f \mid f: X \to \mathbb{F}\}$
  Example; $F[0,1], C[0,1]$ over $\mathbb{R}$
- Matrix Space $M_{m\times n}(\mathbb{F})$, $M_{m\times n}(\mathbb{F}) = \left\{A = \left[a_{ij}\right]_{m\times n} : a_{ij} \in \mathbb{F}\right\}$
- Field over a subfield, example; $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$

## SUBSPACE

A subset $W$ of a vector space $V$ over $\mathbb{F}$ is said to be a subspace if $W$ is itself a vector space over the field $\mathbb{F}$.

### EXAMPLE

- The polynomial space $P_n(X)$ is a subspace of Polynomial subspace $P(X)$ over $\mathbb{F}$
- The polynomial space $P(X)$ is a subspace of function space $F(X)$ over $\mathbb{F}$
- The set of all even functions on the set $X$ form a subspace of the function space $F(X)$ over $\mathbb{F}$
- The set of all symmetric/skew-symmetric matrices form a subspace of the space $M_n(\mathbb{F})$ over $\mathbb{F}$.

### NOTE

- If $W_1 \& W_2$ are subspace of $V$, then so are $W_1 \cap W_2$, $W_1 + W_2$.
- If $W_1 \& W_2$ are subspace of $V$, then $W_1 \cup W_2$ is subspace of $V$ iff one is contained in another.

## LINEAR INDEPENDENCE

Let $V$ be a vector space over $\mathbb{F}$, then a linear combination of vectors $v_1, v_2, \dots, v_n$ in $V$ is a vector

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ for some scalars $c_1, c_2, \dots, c_n$ in $\mathbb{F}$.

The vectors $v_1, v_2, \dots, v_n$ are said to be linearly independent

if $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \Rightarrow c_i = 0$, for every $i$, otherwise the vectors are said to be linearly dependent.

### NOTE

- The set $\{0\}$ is linearly dependent
- The set $\{u\}, u \neq 0$, is linearly independent.
- The set $\{v_1, v_2, \dots, v_n\}$ is linearly dependent if any of $v_i$ is equal to $0$.
- The set $\{v_1, v_2, \dots, v_n\}$ is linearly dependent if $v_i = cv_j$ for some $i \neq j, c \in \mathbb{F}$.
- Any subset of a linearly independent set is linearly independent.
- Any superset of a linearly dependent set is linearly dependent
- The empty set $\phi$ is linearly independent.

## SPANNING SET

Let $V$ be a vector space over $\mathbb{F}$, then the subset $S = \{v_1, v_2, \dots, v_n\}$ of $V$ if for any $v \in V$,

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ for some scalars $c_1, c_2, \dots, c_n$ in $\mathbb{F}$.

## LINEAR SPAN OF A SET

Let $V$ be a vector space over $\mathbb{F}$, $S = \{v_1, v_2, \dots, v_n\}$ be a subset of $V$, then the Linear span $L(S)$ of $S$ is given by $L(S) = \{v \in V : v = c_1v_1 + c_2v_2 + \cdots + c_nv_n, c_i \in \mathbb{F}\}$

### NOTE

For any subset $S \neq \phi$ of a vector space $V$ over $\mathbb{F}$, $L(S)$ is a subspace of $V$.

## BASIS OF A VECTOR SPACE

A subset $S$ of a vector space $V$ over $\mathbb{F}$, is a basis for $V$ if it is linearly independent and spans $V$.

### EXAMPLE

Then vectors $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ constitute a basis for the vector space $\mathbb{R}^3$.

## DIMENSION OF A VECTOR SPACE

A vector space $V$ is finite dimensional if it has a finite basis its dimension is the no. of elements in the basis. If $V$ does not have a finite basis it is infinite dimensional.

### NOTE

- Any subset of a vector space $V$ (of dimension $n$) having more than $n$ vectors is linearly independent.
- Let $S$ be a subset of a vector space $V$ (of dimension $n$) having $n$ vectors, is linearly independent iff $L(S) = V$
- Let $W$ be a subspace of a finite dimensional space $V$, then $\dim W \leq \dim V$
- Let $V$ be a vector space over $\mathbb{F}$ with $\dim V = n$ and $\{v_1, v_2, \dots, v_n\}$ be a basis for $V$, if $A \in GL_n(\mathbb{F})$, then the set $\{Av_1, Av_2, \dots, Av_n\}$ also constitute a basis for $V$.
- Let $V$ be a vector space over $\mathbb{F}$ with $\dim V = n$ and $\mathbb{F}$ is a finite field, then $V$ is a finite vector space.
- Let $A \in M_n(\mathbb{F})$ and let $W = L\{I, A, A^2, \dots, A^n, \dots\}$, then $\dim W \leq n$.
- Let $V$ be a V.S over $\mathbb{F}$ has dimension $n$ and $|\mathbb{F}| = p^k$, for some prime $p$, then $|V| = p^{nk}$

## SUM AND DIRECT SUM

$W_1$ and $W_2$be two subspaces of an $\mathbb{F}$-vector space $V$, then $W_1 + W_2$ is the subspace which is given by

$W_1 + W_2 = \{u + v : u \in W_1, v \in W_2\}$.

$V$ is the direct sum of the spaces $W_1$ and $W_2$ denoted by $V = W_1 \oplus W_2$ if

1. $W_1 + W_2 = V$
2. $W_1 \cap W_2 = \{0\}$

### EXAMPLE

$\mathbb{R}^2$ is direct sum of $x$-axis and $y$-axis.

### NOTE

- If $W_1 \& W_2$ are subspaces of $V$, then $L(W_1 \cup W_2) = W_1 + W_2$.
- $\dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$
- $W_1 \cap (W_2 + W_3) \supseteq (W_1 \cap W_2) + (W_1 \cap W_3)$
- $W_1 + (W_2 \cap W_3) \subseteq (W_1 + W_2) \cap (W_1 + W_3)$
- $\dim(W_1 + W_2 + W_3) = \dim W_1 + \dim W_2 + \dim W_3 - \dim(W_1 \cap W_3) - \dim\big(W_1 \cap (W_2 + W_3)\big)$
- $\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$
- $B_1$ and $B_2$ are bases for $W_1$ and $W_2$ respectively $\Rightarrow B_1 \cup B_2$ is a basis for $W_1 \oplus W_2$.

# LINEAR TRANSFORMATION

## LINEAR TRANSFORMATION

Let V and W be two vector spaces over same field F. Then the map $T: V \to W$ is said to be a linear transformation if

1. $T(v_1 + v_2) = T(v_1) + T(v_2) \forall v_1, v_2 \in V$
2. $T(\alpha v) = \alpha T(v) \forall \alpha \in F and v \in V$

## LINEAR MAP

Let $V$ and $W$ be two V.S over $\mathbb{F}$, a map $T : V \to W$ is said to be a linear map if $T(cv_1 + v_2) = cTV_1 + Tv_2,\ v_1, v_2 \in V$ and $c \in \mathbb{F}$.

## KERNEL OF A LINEAR TRANSFORMATION

$N(T) = Ker(T) = \{v \in V : Tv = 0\}$

- $Ker(T)$ is a subspace of $V$ and dim $Ker(T)$is the nullity $n(T)$ of $T$.
- Let $T: V \to V$ be a linear transformation, then $Ker(T) \subseteq Ker(T^2) \subseteq \cdots$ and $n(T) \leq n(T^2) \leq \cdots$

## RANGE SPACE

Let $T: V \to W$ be a linear transformation, then $Im(T) = R(T) = \{Tv : v \in V\}$

- $R(T)$ is a subspace of $W$ and dim $R(T)$ is the rank $\rho(T)$ of $T$

## RANK-NULLITY THEOREM

$$\rho(T) + n(T) = \dim V$$

- Let $T: V(\mathbb{F}) \to W(\mathbb{F})$ be linear, then $\{v_1, v_2, \ldots, v_n\}$ spans $V \Rightarrow \{Tv_1, \ldots, Tv_n\}$ spans $R(T)$

- Let $\dim V = \dim W = n$, and Let $B = \{v_1, v_2, \dots, v_n\}$ is basis for $V$ and $B' = \{w_1, w_2, \dots, w_n\}$ is for $W$, then $\exists\, T: V \to W$ such that $Tv_1 = w_1, Tv_2 = w_2, \dots, Tv_n = w_n$.
- $T: V \to W$, non singular (Injective) $\Leftrightarrow Ker(T) = \{0\}$, here $\dim V = \dim R(T)$
- $T: V \to W$, $\dim V = \dim W, T$ is injective $\Leftrightarrow$ surjective.

## MATRIX REPRESENTATION OF LINEAR TRANSFORMATION

Let $V$ and $W$ be two vector spaces with $\dim V = n$ and $\dim W = m$, let $B = \{v_1, v_2, \dots, v_n\}$ and $B' = \{w_1, w_2, \dots, w_m\}$ be respective bases.

$$Tv_1 = c_{11}w_1 + c_{21}w_2 + \cdots + c_{m1}w_m$$

$$Tv_2 = c_{12}w_1 + c_{22}w_2 + \cdots + c_{m2}w_m$$

.

.

$$Tv_n = c_{1n}w_1 + c_{2n}w_2 + \cdots + c_{mn}w_m$$

$$[T]_{B'}^{B} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ . & . & . & . & . \\ . & . & . & & . \\ . & . & . & . & . \\ c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mn} \end{pmatrix}$$

- Column vectors are coordinate vectors of $Tv_i$.

## MATRIX TRANSFORMATION

- $A \in M_n(\mathbb{R}), T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Tx = Ax$, then $[T]_B \sim A$for any basis of $\mathbb{R}^n$, and $[T]_B = A \Leftrightarrow B$ is the std. basis of $\mathbb{R}^n$.
- $A \in M_n(\mathbb{R}), T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ define by $TX = AX$, then $m_A(x) = m_T(x)$ (eigen values of $T$ are equal to that of $A$ each with $a.m = n$)
- $T$ is Diagonalizable $\Leftrightarrow A$ is diagonalizable.
- $\Delta_T(x) = \left(\Delta_A(x)\right)^n$
- Let $T$ be a linear operator on a finite dimensional vector space $V$, then
  $$V = Ker(T) \oplus R(T)$$

# CHARECTERISTIC VALUES

## CHARECTERISTIC POLYNOMIAL

Let $A \in M_n(\mathbb{F})$, where $\mathbb{F}$ is a field, then the Characteristic Polynomial $\Delta_A(x)$ of $A$ is given by

$\Delta_A(x) = \det(\lambda I - A)$

## NOTE

- For $A \in M_3(\mathbb{F})$, $\Delta_A(x) = x^2 - \big(trace(A)\big)x + \det A$
- For $A \in M_3(\mathbb{F})$, $\Delta_A(x) = x^3 - \big(trace(A)\big)x^2 + S_2x - \det A$
- For a triangular matrix $A \in M_n(\mathbb{F})$ with diagonal entries $a_{ii}$, then $\Delta_A(x) = \prod_{i=1}^{n}(x - a_{ii})$.

## MINIMAL POLYNOMIAL

let $A \in M_n(\mathbb{F})$, then the minimal polynomial $m_A(x)$ of $A$ is the largest degree monic polynomial satisfied by $A$.

- $m_a(x)$ divides $\Delta_A(x)$.
- $m_a(x)$ & $\Delta_A(x)$ have same irreducible factors.
- $J = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Rightarrow \Delta_j(x) = \Delta_A(x) \cdot \Delta_B(x)$ , $(A, B$ are square matrices.)
- $J = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \Rightarrow m_j(x) = lcm\big(m_A(x), m_c(x)\big)$ , (A,C are square matrices)

## EIGEN VALUES AND EIGEN VECTORS

Let $A \in M_n(\mathbb{F})$ and $\Delta_A(x)$ be its char. Polynomial, then the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of $\Delta_A(x)$ is called the eigen values of $A$.

- **Eigen vector:**
  If $\exists v (\neq 0) \in \mathbb{F}^n$ such that $Av = xv$, then $x$ is an eigen value of $A$ and $v$ is an eigen vector corr. to the eigen value $x$.
- **Algebraic Multiplicity (a.m)**
  a.m of the eigen value $x$ of $A$ is the multiplicity of $x$, as a root in the equation
  $\det(xI - A) = 0$
- **Eigen Space**
  Corresponding to the eigen value $x$ of $A$, $E_x = \{v : (A - xI)v = 0\}$. i.e, $E_x$ is the solution space of the system $(A - xI)v = 0$.
- **Geometric multiplicity (g.m)**
  g.m of $x = \dim E_x$.

## NOTE

let $A \in M_n(\mathbb{F})$,

- For any eigen value $x$ of $A$, $1 \leq g.m(x) \leq a.m(x)$
- If $v_1, v_2$ are eigen vectors of $A$ corr. to the eigen value $x$ then $c_1v_1 + c_2v_2,\ c_1, c_2 \in \mathbb{F}$ (providing $c_1v_1 + c_2v_2 \neq 0$) is again an eigen vector of $A$ corr. to $x$.
- $v$ is an eigen vector of $A$ corr. to $x \iff kv$ is an eigen vector of $A$ corr. to $x$, for any scalar $k$.
- $g.m(x) = n - \rho(A - xI)$ = no. of lin. Indpt. Eigen vectors corr. to $x$.

- $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigen values of $A \in M_n(\mathbb{F})$ and $v_1, v_2, \dots, v_k$ are corr. eigen vectors, then the set $\{v_1, v_2, \dots, v_k\}$ is linearly independent.
- If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of $A \in M_3(\mathbb{F})$,
  $\Delta_A(x) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1.\lambda_2 + \lambda_2\lambda_3 + \lambda_1.\lambda_3)x - \lambda_1.\lambda_2.\lambda_3$
- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of $A \in M_n(\mathbb{F})$, need not be distinct, then
  $trace(A) = \sum_{i=1}^{n} \lambda_i$ & $\det A = \prod_{i=1}^{n} \lambda_i$
- Eigen values of triangular matrices are exactly the diagonal entries.

### CAYLEY HAMILTON THEOREM

Every square matrix satisfies its characteristic equation, i.e, $\Delta_A(x) = 0$

# DIAGONALIZATION

### SIMILAR MATRICES

Let $A, B \in M_n(\mathbb{R})$, then $A$ is similar to $B$, denoted by $A \sim B$, if there exist an invertible $P$ such that $P^{-1}AP = B$

- $A \sim B \Rightarrow \rho(A) = \rho(B)$
- $A \sim B \Rightarrow \Delta_A(x) = \Delta_B(x)$
- $A \sim B \Rightarrow \det A = \det B$
- $A \sim B \Rightarrow m_A(x) = m_B(x)$
- $A \sim B \Rightarrow Trace(A) = Trace(B)$
- $A \sim B^{-1}AB$, for any invertible $B$
- $A, B \in M_n(\mathbb{F}) \Rightarrow$ eigen values of $AB$ and $BA$ are equal $\Rightarrow Trace(BA) = Trace(AB)$ and $\det BA = \det AB$
- $A \sim B \Rightarrow A = PBP^{-1}$, Suppose $x$ is an eigen vector of $A$ corresponding to the eigen value $\lambda$, then $P^{-1}x$ is an eigen vector of $B$ corresponding value $\lambda$.
- $A$ is congruent to $B$, if there exist an invertible $P$ such that $P^t AP = B$
- $A, B \in M_n(\mathbb{R})$ with any one of $A$ or $B$, invertible
  $\Rightarrow AB = BA$
  $\Longrightarrow \Delta_{BA}(x) = \Delta_{AB}(x)$
  $\Rightarrow m_{BA}(x) = m_{AB}(x)$
  $\Rightarrow \rho(BA) = \rho(AB)$
- Let $A, B \in M_n(\mathbb{R})$ and both are not invertible, then $\rho(AB)$ need not be equal to $\rho(BA)$

### DIAGONALIZABILITY

$A \in M_n(\mathbb{F})$ is said to be diagonalizable if $\exists P$ such that $P^{-1}AP = D$, D is a diagonal matrix

### NOTE

- $A$ is diagonalizable $\Leftrightarrow m_A(x)$ is a product of distinct linear factors.
- $a.m(x) = g.m(x)$, for all eigen values $x$ of $A \Leftrightarrow A$ is Diagonalizable
- $\exists\ n$ linearly independent eigen vectors for $A \ \Leftrightarrow A$ is diagonalizable
- If all eigen values of $A$ are distinct $\Rightarrow A$ is diagonalizable

- Zero matrix is diagonalizable over any field.
- Diagonal matrices are diagonalizable over any field
- Scalar matrices are diagonalizable over any field
- Idempotent matrices are diagonalizable over any field
- Involuntary matrices are diagonalizable over any field $F$ with $char. F \neq 2$
  Non-zero nilpotent matrices are not diagonalizable over any field
- Real symmetric matrices are diagonalizable over $\mathbb{R}$
- Hermitian matrices are diagonalizable over $\mathbb{C}$
- Rank 1 matrices are diagonalizable over any field
- $A^3 = A \Rightarrow A$ is diagonalizable over $R$
- $A^3 = I \Rightarrow A$ is diagonalizable over $C$
- $A^3 = -A \Rightarrow A$ is diagonalizable over $C$
- $A^4 = I$ but $A^2 \neq I \Rightarrow A$ is diagonalizable only over $\mathbb{C}$
- Orthogonal matrices are diagonalizable over $\mathbb{C}$
- Normal matrices are diagonalizable over $\mathbb{C}$

# HYPER SPACE

## HYPER SPACE

Let $V$ be a finite dimensional vector space with $\dim V = n$, then any subspace $W$ of $V$ with dimension $n - 1$ is called a hyper space of $V$.

## QUOTIENT SPACE

Let $V$ be a V.S over $\mathbb{F}$ and $W$ be a subspace of $V$, then the quotient space $V/W$ is the space given by $V/W = \{v + W : v \in V\}$

$\dim V/W = \dim V - \dim W$