CHARGED PARTICLE MOTION IN CONSTANT AND UNIFORM ELECTROMAGNETIC FIELDS

1. INTRODUCTION

In this and in the following two chapters we investigate the motion of charged particles in the presence of electric and magnetic fields known as functions of position and time. Thus, the electric and magnetic fields are assumed to be prescribed and are not affected by the charged particles. This chapter, in particular, considers the fields to be constant in time and spatially uniform. This subject is considered in some detail, since many of the more complex situations, considered in Chapters 3 and 4, can be treated as perturbations to this problem.

The study of the motion of charged particles in specified fields is important, since it provides a good physical insight for understanding some of the dynamical processes in plasmas. It also permits to obtain information on some macroscopic phenomena which are due to the collective behavior of a large number of particles. Not all of the components of the detailed microscopic particle motion contribute to macroscopic effects, but it is possible to isolate the components of the individual motion that contribute to the collective plasma behavior. Nevertheless, the macroscopic parameters can be obtained much more easily and conveniently from the macroscopic transport equations presented in Chapters 8 and 9.

The equation of motion for a particle of charge q, under the action of the Lorentz force $\mathbf{F}$ due to electric ($\mathbf{E}$) and magnetic induction ($\mathbf{B}$) fields, can be written as

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.1}$$

where $\mathbf{p}$ represents the momentum of the particle and $\mathbf{v}$ its velocity. This equation is relativistically correct if we take

$$\mathbf{p} = \gamma m \mathbf{v} \tag{1.2}$$

where m is the rest mass of the particle and γ is the so-called *Lorentz factor* defined by

$$\gamma = (1 - v^2/c^2)^{-1/2} \tag{1.3}$$

where c is the speed of light in vacuum. In the relativistic case, (1.1) can also be written in the form

$$\gamma m \frac{d\mathbf{v}}{dt} + q\left(\frac{\mathbf{v}}{c^2}\right)(\mathbf{v} \cdot \mathbf{E}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.4}$$

noting that the time rate of change of the total relativistic energy ($U = \gamma mc^2$) is given by $dU/dt = q(\mathbf{v}\cdot\mathbf{E})$ and that $d\mathbf{p}/dt = d(U\mathbf{v}/c^2)/dt$.

In many situations of pratical interest, however, the term v^2/c^2 is neglibible compared to unity. For $v^2/c^2 \ll 1$ we have $\gamma \simeq 1$ and m can be considered constant (independent of v), so that (1.4) reduces to the following nonrelativistic expression

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.5}$$

If the velocity obtained from (1.5) does not satisfy the condition $v^2 \ll c^2$, then the corresponding result is not valid and the relativistic expression (1.4) must be used instead of (1.5). Relativistic effects become important only for highly energetic particles (a 1 MeV proton, for instance, has a velocity of 1.4×10^7 m/s, with $v^2/c^2 \simeq 0.002$). For the situations to be considered here it is assumed that the restriction $v^2 \ll c^2$, implicit in (1.5), is not violated. Also, all radiation effects are neglected.

2. ENERGY CONSERVATION

In the absence of an electric field (**E** = 0), the equation of motion (1.5) reduces to

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}) \tag{2.1}$$

Since the magnetic force is perpendicular to **v**, it does no work on the particle. Taking the dot product of (2.1) with **v** and noting that for any vector **v**, we have (**v** × **B**) · **v** = 0, we obtain

$$m\frac{\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) \; d = 0 \tag{2.2}$$

which shows that the particle kinetic energy ($mv^2/2$) and the magnitude of its velocity (speed v) are both constants . Therefore, a *static* magnetic field does not change the particle kinetic energy. This result is valid whatever the spatial dependence of the magnetic flux density **B**. However, if **B** varies with time then, according to Maxwell equations, an electric field such that $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ is also present which does work on the particle changing its kinetic energy.

When both *magnetostatic* and *electrostatic* fields are present, we obtain from (1.5)

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = q(\mathbf{E} \cdot \mathbf{v}) \tag{2.3}$$

Since $\nabla \times \mathbf{E} = 0$, we can express the electrostatic field in terms of the electrostatic potential according to $\mathbf{E} = -\nabla\varphi$, so that

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = -q(\nabla\phi) \cdot \mathbf{v} = -q(\nabla\phi) \cdot \frac{d\mathbf{r}}{dt} = -q\frac{d\phi}{dt} \tag{2.4}$$

This result can be rearranged in the following conservation form

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2 + q\phi\right) = 0 \tag{2.5}$$

which shows that the sum of the particle kinetic and electric potential energies remains constant in the presence of *static* electromagnetic fields. Note that the electric potential φ can be considered as the potential energy per unit charge.

When the fields are time-dependent we have $\nabla \times \mathbf{E} \neq 0$ and **E** is not the gradient of a scalar function. But, since $\nabla \cdot \mathbf{B} = 0$, we can define a magnetic vector potential **A** by $\mathbf{B} = \nabla \times \mathbf{A}$ and write (1.5.2) as

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} + \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \qquad (2.6)$$

Hence, we can express the electric field in the form

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad (2.7)$$

In this case the system is not conservative in the usual sense and there is no energy integral, but the analysis may be performed using a *Lagrangian function L* for a charged particle in electromagnetic fields, defined by $L = \frac{1}{2}mv^2 - U$ (2.8)

where U is a velocity-dependent potential energy given by

$$U = q(\varphi - \mathbf{v} \cdot \mathbf{A}) \qquad (2.9)$$

The energy considerations presented in this section assume that the particle energy changes only as a result of the work done by the fields. This assumption is not strictly correct since every charged particle when accelerated irradiates energy in the form of electromagnetic waves. For the situations to be considered here this effect is usually small and can be neglected.

3. UNIFORM ELECTROSTATIC FIELD

According to (1.1) the motion of a charged particle in an electric field obeys the following differential equation

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} \qquad (3.1)$$

For a constant **E** field, (3.1) can be integrated directly giving

$$\mathbf{p}(t) = q\mathbf{E}t + \mathbf{p}_o \qquad (3.2)$$

where $\mathbf{p}_o = \mathbf{p}(0)$ denotes the initial particle momentum. Using the nonrelativistic expression

$$\mathbf{p} = m\mathbf{v} = m\frac{d\mathbf{r}}{dt} \quad (3.3)$$

and performing a second integration in (3.2), we obtain the following expression for the particle position as a function of time

$$\mathbf{r}(t) = \frac{1}{2}\left(\frac{q\mathbf{E}}{m}\right)t^2 + \mathbf{v}_o t + \mathbf{r}_o \quad (3.4)$$

where $\mathbf{r}_o$ denotes the particle initial position and $\mathbf{v}_o$ its initial velocity. Therefore, the particle moves with a constant acceleration, $q\mathbf{E}/m$, in the direction of $\mathbf{E}$ if $q > 0$, and in the opposite direction if $q < 0$. In a direction perpendicular to the electric field there is no acceleration and the particle state of motion remains unchanged.

4. UNIFORM MAGNETOSTATIC FIELD

4.1 Formal Solution of the Equation of Motion

For a particle of charge q and mass m, moving with velocity $\mathbf{v}$ in a region of space where there is only a magnetic induction $\mathbf{B}$ (no electric field $\mathbf{E}$), the equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}) \quad (4.1)$$

It is convenient to separate $\mathbf{v}$ in components parallel ($\mathbf{v}$) and perpendicular ($\mathbf{v}_\perp$) to the magnetic field,

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \quad (4.2)$$

as indicated in Fig. 1. Substituting (4.2) into (4.1) and noting that ($\mathbf{v}_\parallel \times \mathbf{B} = 0$) we obtain

$$\frac{d\mathbf{v}_\parallel}{dt} + \frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) \quad (4.3)$$

Since the term ($\mathbf{v}_\perp \times \mathbf{B}$) is perpendicular to **B**, the *parallel* component equation can be written as

$$\frac{d\mathbf{v}_\parallel}{dt} = 0 \tag{4.4}$$

and the *perpendicular* component equation as

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) \tag{4.5}$$

Eq. (4.4) shows that the particle velocity *along* **B** does not change and is equal to the particle initial velocity. For motion in the plane *perpendicular* to **B**, we can write (4.5) in the form

$$\frac{d\mathbf{v}_\perp}{dt} = \mathbf{\Omega}_c \times \mathbf{v}_\perp \tag{4.6}$$

where $\mathbf{\Omega}_c$ is a vector defined by

$$\mathbf{\Omega}_c = -\frac{q\mathbf{B}}{m} = \frac{|q|\ B}{m}\widehat{\mathbf{\Omega}}_c = \Omega_c\widehat{\mathbf{\Omega}}_c \tag{4.7}$$

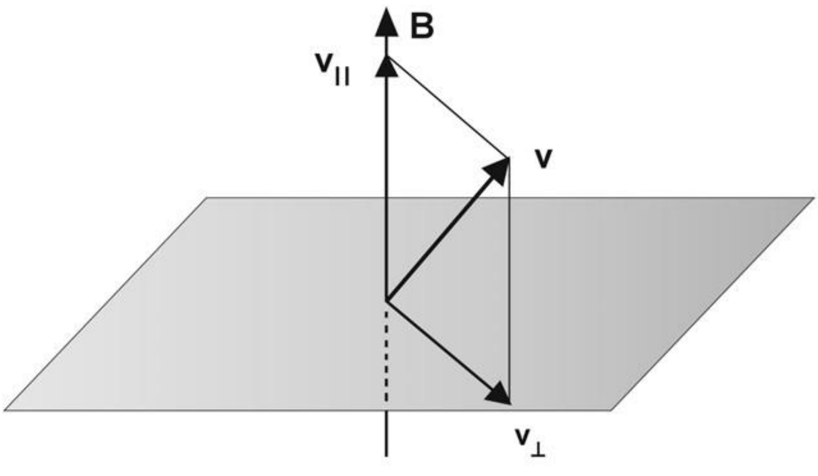

Fig. 1 Decomposition of the velocity vector into components parallel (**v**) and perpendicular ($\mathbf{v}_\perp$) to the magnetic field.

Thus, $\widehat{\mathbf{\Omega}}_c$ points in the direction of **B** for a *negatively* charged particle

ENTRI

($q < 0$) and in the opposite direction for a *positively* charged particle ($q >$ 0). Its magntitude Ω_c is always positive ($\Omega_c = |q|\ B/m$). The unit vector $\mathbf{\Omega}_c$ points along $\mathbf{\Omega}_c$.

Since $\mathbf{\Omega}_c$ is constant and, from conservation of kinetic energy, $v_\perp$ (the magnitude of $\mathbf{v}_\perp$) is also constant, (4.6) shows that the particle acceleration is constant in magnitude and its direction is perpendicular to both $\mathbf{v}_\perp$ and $\mathbf{B}$. Thus, this acceleration corresponds to a rotation of the velocity vector $\mathbf{v}_\perp$ in the plane perpendicular to $\mathbf{B}$ with the constant angular velocity $\mathbf{\Omega}_c$. We can integrate (4.6) directly, noting that $\mathbf{\Omega}_c$ is constant and taking $\mathbf{v}_\perp = d\mathbf{r}_c/dt$, to obtain

$$\mathbf{v}_\perp = \mathbf{\Omega}_c \times \mathbf{r}_c \tag{4.8}$$

where the vector $\mathbf{r}_c$ is interpreted as the particle position vector with respect to a point G (the center of gyration) in the plane perpendicular to $\mathbf{B}$ which contains the particle. Since the particle speed $v_\perp$ is constant, the magnitude r_c of the position vector is also constant. Therefore, (4.8) shows that the velocity $\mathbf{v}_\perp$ corresponds to a rotation of the position vector $\mathbf{r}_c$ about the point G in the plane perpendicular to $\mathbf{B}$ with constant angular velocity $\mathbf{\Omega}_c$. The component of the motion in the plane perpendicular to $\mathbf{B}$ is therefore a circle of radius r_c. The *instantaneous* center of gyration of the particle (the point G at the distance r_c from the particle) is called the

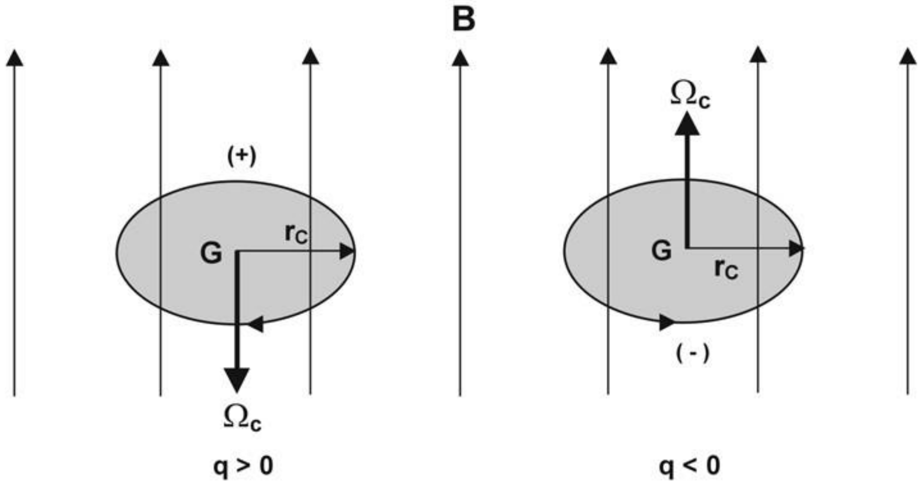

Fig. 2 Circular motion of a charged particle about the guiding center in a uniform magnetostatic field.

guiding center. This circular motion about the guiding center is illustrated in Fig. 2.

Note that according to the definition of $\mathbf{\Omega}_c$, given in (4.7), $\hat{\mathbf{\Omega}}_c$ always points in the same direction as the particle *angular momentum vector* ($\mathbf{r}_c \times \mathbf{p}$), irrespective of its charge.

The resulting trajectory of the particle is given by the superposition of a *uniform* motion *along* **B** (with the constant velocity **v**) and a *circular* motion in the plane *normal* to **B** (with the constant speed $v_\perp$). Hence, the particle describes a *helix* (see Fig. 3). The angle between **B** and the direction of motion of the particle is called the *pitch angle* and is given by

$$\alpha = \sin^{-1}\left(\frac{v_\perp}{v}\right) = \tan^{-1}\left(\frac{v_\perp}{v_\parallel}\right) \quad (4.9)$$

where v is the total speed of the particle ($v^2 = v_\parallel^2 + v_\perp^2$). When $v_\parallel = 0$ but $v_\perp \neq 0$, we have $\alpha = \pi/2$ and the particle trajectory is a circle in the plane normal to **B**. On the other hand, when $v_\perp = 0$ but $v_\parallel \neq 0$, we have $\alpha = 0$ and the particle moves along **B** with the velocity **v**.

The magnitude of the angular velocity,

$$\Omega_c = \frac{|q|\ B}{m} \tag{4.10}$$

is known as the *angular frequency of gyration*, and is also called the *gyrofrequency*, or *cyclotron frequency* or *Larmor frequency*. For an electron $|q| = 1.602 \times 10^{-19}$ coulomb and $m = 9.109 \times 10^{-31}$ kg, so that

$$\Omega_c(electron) = 1.76 \times 10^{11} B \qquad \text{(rad/s)} \tag{4.11}$$

with B in tesla (or, equivalently, weber/m^2). Similarly, for a proton $m = 1.673 \times 10^{-27}$ kg, so that

$$\Omega_c(proton) = 9.58 \times 10^{7} B \qquad \text{(rad/s)} \tag{4.12}$$

The radius of the circular orbit, given by

$$r_c = \frac{v_\perp}{\Omega_c} = \frac{m v_\perp}{|q|\ B} \tag{4.13}$$

is called the *radius of gyration*, or *gyroradius*, or *cyclotron radius*, or *Larmor radius*. It is important to note that Ω_c is directly proportional to B. Consequently, as B increases, the gyrofrequency increases and the radius decreases. Also, the smaller the particle mass the larger will be its gyrofrequency and the smaller its gyroradius. Multiplying (4.13) by B gives

$$B r_c = \frac{m v_\perp}{|q|} = \frac{p_\perp}{|q|} \tag{4.14}$$

which shows that the magnitude of B times the particle gyroradius is equal to the particle momentum per unit charge. This quantity is often called the *magnetic rigidity*.

4.2 Solution in Cartesian Coordinates

The treatment presented so far in this section was not related to any particular frame of reference. Consider now a Cartesian coordinate system (x,y,z) such that $\mathbf{B} = B\hat{\mathbf{z}}$. In this case, the cross product between $\mathbf{v}$ and $\mathbf{B}$ can be written as

$$\mathbf{v} \times \mathbf{B} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = B(v_y \hat{\mathbf{x}} - v_x \hat{\mathbf{y}}) \tag{4.15}$$

and the equation of motion (4.1) becomes

$$\frac{d\mathbf{v}}{dt} = \frac{qB}{m}(v_y\hat{\mathbf{x}} - v_x\hat{\mathbf{y}}) = \pm\Omega_c(v_y\hat{\mathbf{x}} - v_x\hat{\mathbf{y}}) \qquad (4.16)$$

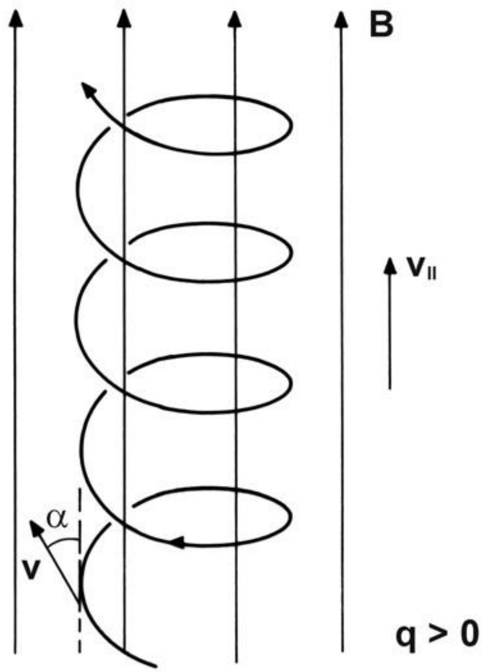

Fig. 3 Helicoidal trajectory of a positively charged particle in a uniform magnetostatic field.

The (+) sign in front of Ω_c applies to a positively charged particle ($q > 0$) and the (−) sign to a negatively charged particle ($q < 0$), since Ω_c is always positive, according to its definition given in (4.10). In what follows we shall consider a positively charged particle. The results for a negative charge can be obtained by changing the sign of Ω_c in the results for the positive charge.

The Cartesian components of (4.16) are (for $q > 0$)

$$\frac{dv_x}{dt} = \Omega_c v_y \qquad (4.17)$$

$$\frac{dv_y}{dt} = -\Omega_c v_x \qquad (4.18)$$

The last of these equations gives $\frac{dv_z}{dt} = 0$ (4.19)

$v_z(t) = v_z(0) = v_\parallel$,

which is the initial value of the velocity component parallel to **B**. To obtain the solution of (4.17) and (4.18), we take the derivative of (4.17) with respect to time and substitute this result into (4.18), getting

$$\frac{d^2 v_x}{dt^2} + \Omega_c^2 v_x = 0 \tag{4.20}$$

This is the homogeneous differential equation for a *harmonic oscillator* of frequency Ω_c, whose solution is

$$v_x(t) = v_\perp \sin(\Omega_c t + \theta_o) \tag{4.21}$$

where $v_\perp$ is the constant speed of the particle in the (x,y) plane (normal to **B**) and θ_o is a constant of integration which depends on the relation between the initial velocities $v_x(0)$ and $v_y(0)$, according to

$$\tan(\theta_o) = v_x(0)/v_y(0) \tag{4.22}$$

To determine $v_y(t)$ we substitute (4.21) in the left-hand side of (4.17), obtaining

$$v_y(t) = v_\perp \cos(\Omega_c t + \theta_o) \tag{4.23}$$

Note that $v_x^2 + v_y^2 = v_\perp^2$. The equations for the components of **v** can be further integrated with respect to time, yielding

$$x(t) = -\frac{v_\perp}{\Omega_c} \cos(\Omega_c t + \theta_o) + X_o \tag{4.24}$$

$$y(t) = \frac{v_\perp}{\Omega_c} \sin(\Omega_c t + \theta_o) + Y_o \tag{4.25}$$

$$z(t) = v_\parallel t + z_o \tag{4.26}$$

where we have defined

$$X_o = x_o + \frac{v_\perp}{\Omega_c}\cos(\theta_o) \tag{4.27}$$

$$Y_o = y_o - \frac{v_\perp}{\Omega_c}\sin(\theta_o) \tag{4.28}$$

The vector $\mathbf{r} = x_o\widehat{\mathbf{x}} + y_o\widehat{\mathbf{y}} + z_o\widehat{\mathbf{z}}$ gives the initial particle position. From (4.24) and (4.25) we see that

$$(x - X_o)^2 + (y - Y_o)^2 = (v_\perp/\Omega_c)^2 = r_c^2 \tag{4.29}$$

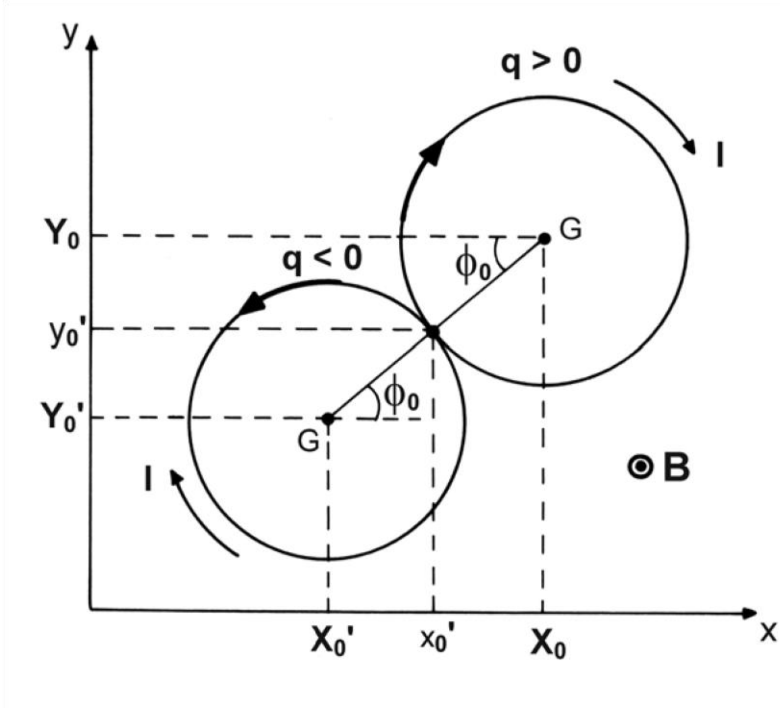

Fig. 4 Circular trajectory of a charged particle in a uniform and constant **B** field (directed out of the paper), and the direction of the associated electric current.

The particle trajectory in the plane normal to **B** is therefore a circle with center at (X_o, Y_o) and radius equal to $(v_\perp/\Omega_c)$. The motion of the point $[X_o, Y_o, z(t)]$, at the instantaneous center of gyration, corresponds to the trajectory of the guiding center. Thus, the guiding center moves with constant velocity v along **B**.

In the (x,y) plane, the argument $\varphi(t)$, defined by

$$\phi(t) = \tan^{-1}\frac{(y - Y_o)}{(x - X_o)} = -(\Omega_c t + \theta_o) ; \qquad \varphi_o = -\theta_o \qquad (4.30)$$

decreases with time for a positively charged particle. For a magnetic field pointing towards the observer, a *positive* charge describes a circle in the *clockwise* direction. For a negatively charged particle Ω_c must be replaced by $-\Omega_c$ in the results of this sub-section. Hence, (4.30) shows that for a *negative* charge $\varphi(t)$ increases with time and the particle moves in a circle in the *counterclockwise* direction, as shown in Fig. 4. The resulting particle motion is a *cylindrical helix* of constant pitch angle. Fig. 5 shows the parameters of the helix with reference to a Cartesian coordinate system.

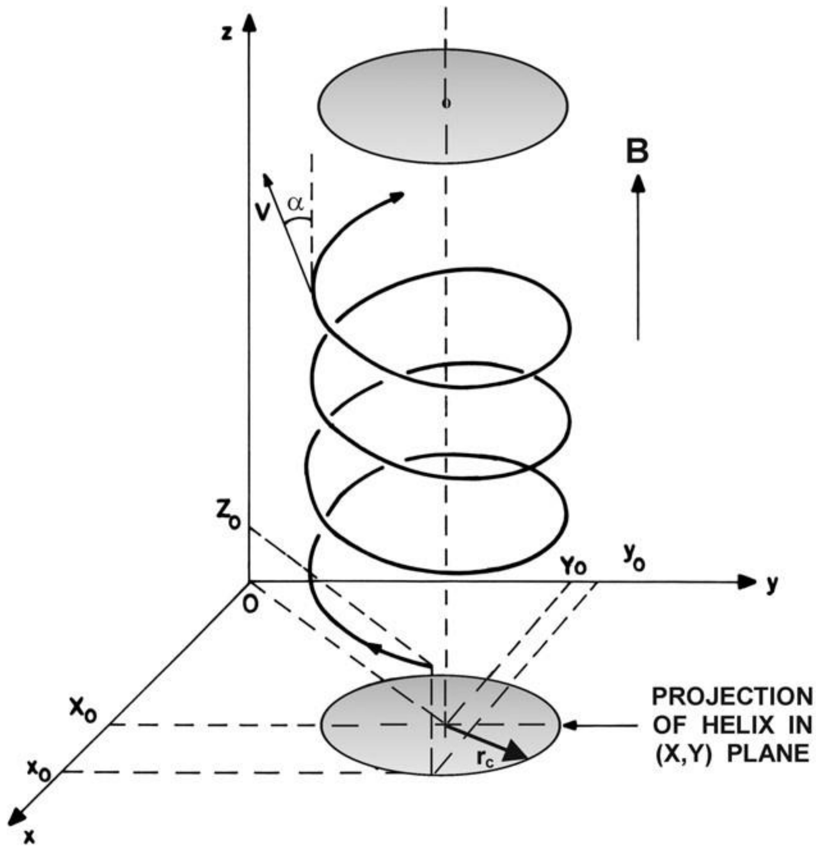

Fig. 5 Parameters of the helicoidal trajectory of a positively charged particle with reference to a Cartesian coordinate system.

4.3 Magnetic Moment

To the circular motion of a charged particle in a magnetic field there is associated a circulating electric current *I*. This current flows in the clockwise direction for a **B** field pointing towards the observer (Fig. 4). From Ampere's law, the direction of the magnetic field associated with this circulating current is given by the *right-hand rule* i.e. with the right thumb pointing in the direction of the current *I*, the right fingers curl in the direction of the associated magnetic field. Therefore, the **B** field produced by the circular motion of a charged particle is *opposite* to the externally applied **B** field *inside* the particle orbit, but in the same direction outside the orbit. The magnetic field generated by the ring current *I*, at distances much larger than r_c, is similar to that of a *dipole* (Fig. 6).

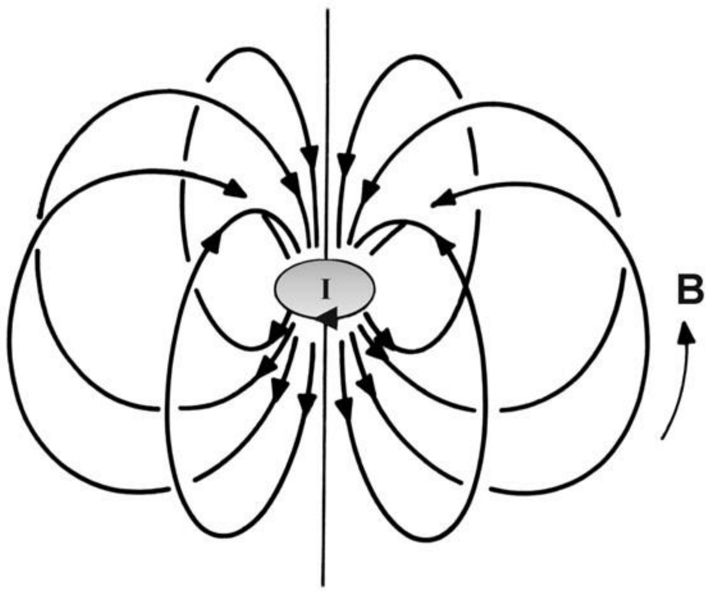

Fig. 6 The magnetic field generated by a small ring current is that of a magnetic dipole.

Since a plasma is a collection of charged particles, it possesses therefore *diamagnetic* properties.

The *magnetic moment* **m** associated with the circulating current is normal to the area *A* bounded by the particle orbit and points in the direction opposite to the externally applied **B** field, as shown in Fig. 7. Its magnitude is given by

$$|\mathbf{m}| = (current) \cdot (orbital\ area) = IA \qquad (4.31)$$

This circulating current corresponds to a flow of charge and is given by

$$I = \frac{|q|}{T_c} = \frac{|q|\ \Omega_c}{2\pi} \qquad (4.32)$$

where $T_c = 2\pi/\Omega_c$ is the period of the particle orbit, known as the *cyclotron period* or *Larmor period*. The magnitude of **m** is therefore

$$|\mathbf{m}| = \frac{|q|\ \Omega_c}{2\pi}\pi r_c^2 = \frac{1}{2}|q|\ \Omega_c r_c^2 \qquad (4.33)$$

Using the relations $\Omega_c = |q|\ B/m$ and $r_c = v_\perp/\Omega_c$, (4.33) becomes

$$|\mathbf{m}| = \frac{\frac{1}{2}mv_\perp^2}{B} = \frac{W_\perp}{B} \qquad (4.34)$$

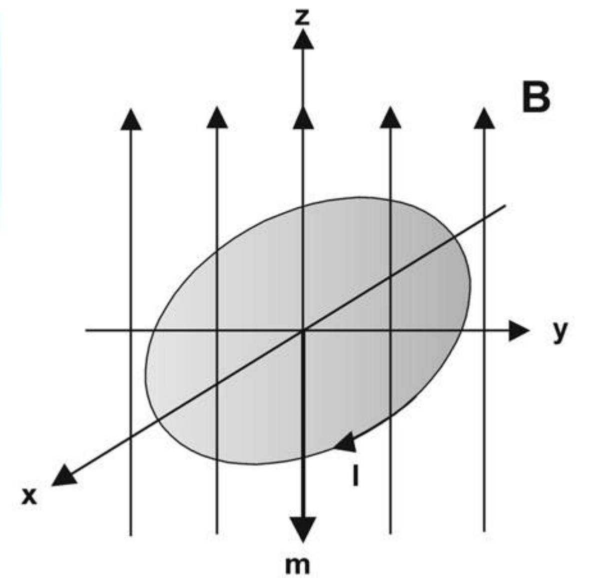

Fig. 7 Magnetic moment **m** associated with a circulating current due to the circular motion of a charged particle in an external **B** field.

where $W_\perp$ denotes the part of the particle kinetic energy associated with the transverse velocity $v_\perp$. Thus, in vector form,

$$\mathbf{m} = \frac{-W_{\perp}\mathbf{B}}{B^2} \tag{4.35}$$

4.4 Magnetization Current

Consider now a collection of charged particles, positive and negative in equal numbers (in order to have no internal macroscopic electrostatic fields), instead of just one single particle. For instance, consider the case of a low-density plasma in which the particle collisions can be neglected (collisionless plasma). The condition for this is that the average time between collisions be much greater than the cyclotron period. This condition is fulfilled for many space plasmas, for example. For a collisionless plasma in an external magnetic field, the magnetic moments due to the orbital motion of the charged particles act together, giving rise to a resultant magnetic field which may be strong enough to appreciably change the externally applied **B** field. The magnetic field produced by the orbital motion of the charged particles can be determined from the *net* electric current density associated with their motion.

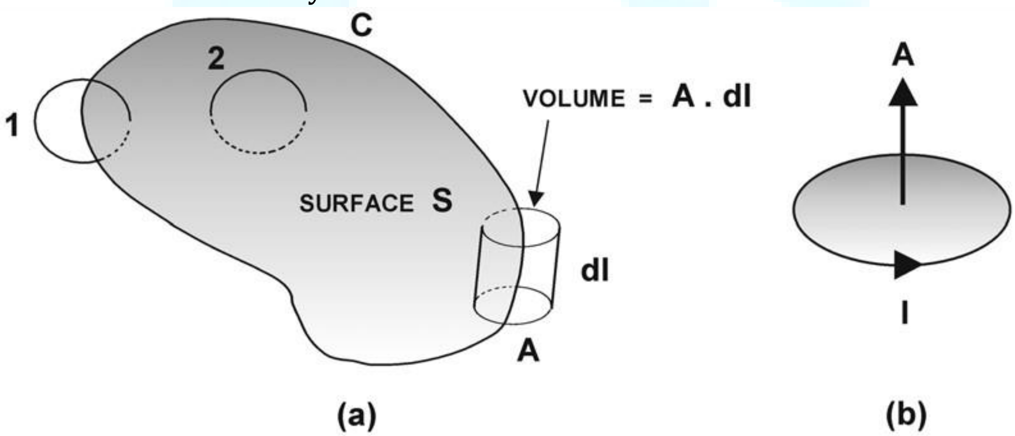

Fig. 8 (a) Electric current orbits crossing the surface element S bounded by the curve C, in a macroscopic volume containing a large number of particles. (b) Positive direction of the vector area **A.**

To calculate the resultant electric current density, let us consider a macroscopic volume containing a large number of particles. Let S be an element of area in this volume, bounded by the curve C, as shown in Fig. 8

(a). Orbits such as (1), which encircle the bounded surface only once, contribute to the resultant current, whereas orbits such as (2), which cross the surface twice, do not contribute to the net current. If $d\mathbf{l}$ is an element of arc along the curve C, the number of orbits encircling $d\mathbf{l}$ is given by $n\mathbf{A} \cdot d\mathbf{l}$, where n is the number of orbits of current I, per unit volume, and $\mathbf{A}$ is the vector area bounded by each orbit. The direction of $\mathbf{A}$ is that of the normal to the orbital area A, the positive sense being related to the sense of circulation in the way the linear motion of a right-hand screw is related to its rotary motion. Thus, $\mathbf{A}$ points in the direction of the observer when I flows counterclockwise, as shown in Fig. 8 (b). The net resultant current crossing S is therefore given by the current encircling $d\mathbf{l}$ integrated along the curve C,

$$I_n = \oint In\mathbf{A} \cdot d\mathbf{l} \tag{4.36}$$

Since $\mathbf{m} = I\mathbf{A}$, the magnetic moment, per unit volume, $\mathbf{M}$, (also called the *magnetization vector*) is given by

$$\mathbf{M} = n\mathbf{m} = nI\mathbf{A} \tag{4.37}$$

Hence, (4.36) can be written as

$$I_n = \oint \mathbf{M} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{M}) \cdot d\mathbf{S} \tag{4.38}$$

where we have applied *Stokes' theorem*. We may define an average *magnetization current density*, $\mathbf{J}_M$, crossing the surface S, by

$$I_n = \int_S \mathbf{J}_M \cdot d\mathbf{S} \tag{4.39}$$

Consequently, from (4.38) and (4.39) we obtain the magnetization current density as

$$\mathbf{J}_M = \nabla \times \mathbf{M} \tag{4.40}$$

where, from (4.37) and (4.35),

$$\mathbf{M} = n\mathbf{m} = -\left(\frac{nW_\perp}{B^2}\right)\mathbf{B} \tag{4.41}$$

and $nW_\perp$ denotes the kinetic energy, per unit volume, associated with the transverse particle velocity.

The charge density ρ_M associated with the magnetization current density $\mathbf{J}_M$ can be deduced from the equation of continuity,

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot \mathbf{J}_M = 0 \tag{4.42}$$

Since $\mathbf{J}_M = \nabla \times \mathbf{M}$ and since for any vector $\mathbf{a}$, we have $\nabla \cdot (\nabla \times \mathbf{a}) = 0$, it follows that the charge density ρ_M is constant.

In the following Maxwell equation

$$\nabla \times \mathbf{B} = \mu_o \Big(\mathbf{J} + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \Big) \tag{4.43}$$

we can separate the *total* current density **J** into two parts: a magnetization current density $\mathbf{J}_M$ and a current density **J** due to other sources,

$$\mathbf{J} = \mathbf{J}_M + \mathbf{J}' \tag{4.44}$$

Expressing $\mathbf{J}_M$ in terms of **M**, through (4.40), and substituting in (4.43), we obtain

$$\nabla \times \mathbf{B} = \mu_o \Big(\nabla \times \mathbf{M} + \mathbf{J}' + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \Big) \tag{4.45}$$

which can be rearranged as

$$\nabla \times \Big(\frac{1}{\mu_o} \mathbf{B} - \mathbf{M} \Big) = \mathbf{J}' + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{4.46}$$

Defining an *effective* magnetic field **H** by the relation

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) \tag{4.47}$$

we can write (4.46) as

$$\nabla \times \mathbf{H} = \mathbf{J}' + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{4..48}$$

Thus, the effective magnetic field **H** is related to the current due to other sources **J**, in the way **B** is related to the total current **J.** Eqs. (4.40) and (4.47) constitute the basic relations for the classical treatment of magnetic materials.

A simple linear relation between **B** and **H** exists when **M** is proportional to **B** or **H**,

$$\mathbf{M} = \chi_m \mathbf{H} \qquad (4.49)$$

where the constant χ_m is called the *magnetic susceptibility* of the medium. However, for a plasma we have seen that $M \propto 1/B$ [see (4.41)], so that the relation between **H** and **B** (or **M**) is not linear. Within this context it is generally not convenient to treat a plasma as a magnetic medium.

5. UNIFORM ELECTROSTATIC AND MAGNETOSTATIC FIELDS

5.1 Formal Solution of the Equation of Motion

We consider now the motion of a charged particle in the presence of both electric and magnetic fields which are constant in time and spatially uniform. The nonrelativistic equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (5.1)$$

Taking components parallel and perpendicular to **B**,

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \qquad (5.2) \qquad \mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp \qquad (5.3)$$

we can resolve (5.1) into two component equations

$$m\frac{d\mathbf{v}_\parallel}{dt} = q\mathbf{E}_\parallel \qquad (5.4)$$

$$m\frac{d\mathbf{v}_\perp}{dt} = q(\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B}) \qquad (5.5)$$

Eq. (5.4) is similar to (3.1) and represents a motion with constant acceleration $q\mathbf{E}_\parallel/m$ along the **B** field. Hence, according to (3.2) and (3.4),

$$\mathbf{v}_\parallel(t) = \left(\frac{q\mathbf{E}_\parallel}{m}\right)t + \mathbf{v}_\parallel(0) \tag{5.6}$$

$$\mathbf{r}_\parallel(t) = \frac{1}{2}\left(\frac{q\mathbf{E}_\parallel}{m}\right)t^2 + \mathbf{v}_\parallel(0)t + \mathbf{r}_\parallel(0) \tag{5.7}$$

To solve (5.5) it is convenient to separate $\mathbf{v}_\perp$ into two components

$$\mathbf{v}_\perp(t) = \mathbf{v}'_\perp(t) + \mathbf{v}_E \tag{5.8}$$

where $\mathbf{v}_E$ is a constant velocity in the plane normal to **B**. Hence, $\mathbf{v}_\perp$ represents the particle velocity as seen by an observer in a frame of reference moving with the constant velocity $\mathbf{v}_E$. Substituting (5.8) into (5.5), and writing the component of the electric field perpendicular to **B** in the form (see Fig. 9)

$$\mathbf{E}_\perp = -\left(\frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}\right) \times \mathbf{B} \tag{5.9}$$

we obtain

$$m\frac{d\mathbf{v}'_\perp}{dt} = q\left(\mathbf{v}'_\perp + \mathbf{v}_E - \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}\right) \times \mathbf{B} \tag{5.10}$$

This equation shows that in a coordinate system moving with the constant velocity

$$\mathbf{v}_E = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2} \tag{5.11}$$

the particle motion in the plane normal to **B** is governed entirely by the magnetic field, according to

$$m\frac{d\mathbf{v}'_\perp}{dt} = q(\mathbf{v}'_\perp \times \mathbf{B}) \tag{5.12}$$

Thus, in this frame of reference, the electric field component $\mathbf{E}_\perp$ is transformed away, whereas the magnetic field is left unchanged. Eq. (5.12) is

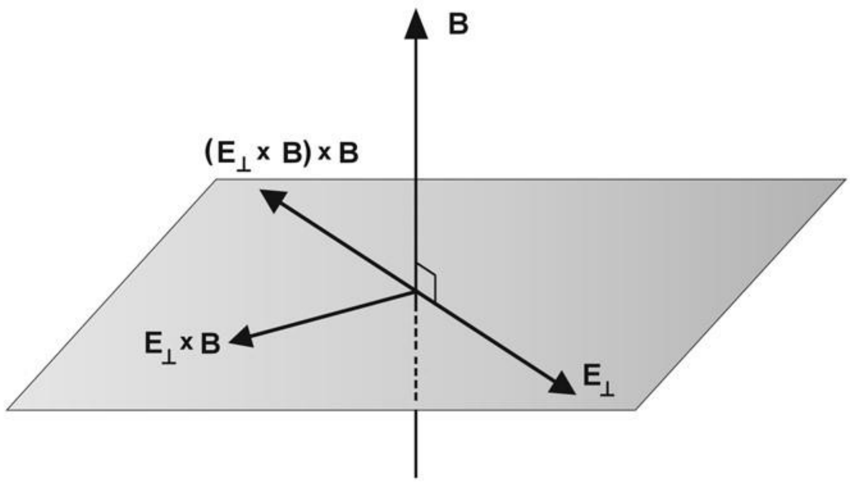

Fig. 9 Vector products appearing in Eq. (5.9) (**B** = **B**/*B*).

identical to (4.5) and implies that in the reference system moving with the constant velocity $\mathbf{v}_E$, given by (5.11), the particle describes a circular motion at the cyclotron frequency Ω_c with radius r_c,

$$\mathbf{v}'_{\perp} = \mathbf{\Omega}_c \times \mathbf{r}_c \tag{5.13}$$

The results obtained so far indicate that the resulting particle motion is described by a superposition of a circular motion in the plane normal to **B**, with a uniform motion with the constant velocity $\mathbf{v}_E$ perpendicular to both **B** and $\mathbf{E}_\perp$, plus a uniform acceleration $q\mathbf{E}_{\parallel}/m$ along **B**. The particle velocity can be expressed in vector form, independently of a coordinate system, as

$$\mathbf{v}(t) = \mathbf{\Omega}_c \times \mathbf{r}_c + \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2} + \frac{q\mathbf{E}_\parallel}{m} t + \mathbf{v}_\parallel(0) \tag{5.14}$$

The first term in the right-hand side of (5.14) represents the cyclotron circular motion, and the following ones represent, respectively, the drift velocity of the guiding center (perpendicular to both $\mathbf{E}_\perp$ and **B**), the constant acceleration of the guiding center along **B**, and the initial velocity parallel to **B**.

Note that the velocity $\mathbf{v}_E$ is independent of the mass and of the sign of the charge and therefore is the same for both positive and negative particles. It is usually called the *plasma drift velocity* or the *electromagnetic plasma drift*. Since $\mathbf{E}_\parallel \times \mathbf{B} = 0$, (5.11) can also be written as

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{5.15}$$

The resulting motion of the particle in the plane normal to **B** is, in general, a *cycloid*, as shown in Fig. 10. The physical explanation for this cycloidal motion is a follows. The electric force $q\mathbf{E}_\perp$, acting simultaneously with the magnetic force, accelerates the particle so as to increase or decrease its velocity, depending on the relative direction of the particle motion with respect to the direction of $\mathbf{E}_\perp$ and on the charge sign. According to (4.13) the radius of gyration increases with velocity and hence the radius of curvature of the particle path varies under the action of $\mathbf{E}_\perp$. This results in a cycloidal trajectory with a net drift in the direction perpendicular to both **E** and **B**. Different trajectories are obtained, depending on the initial conditions and on the magnitude of the applied electric and magnetic fields.

The ions are much more massive than the electrons and therefore the Larmor radius for ions is correspondingly greater and the Larmor frequency correspondingly smaller than for electrons. Consequently, the arcs of cycloid for ions are greater than for electrons, but there is a larger number of arcs of cycloid per second for electrons, such that the drift velocity is the same for both species.

In a collisionless plasma the drift velocity does not imply an electric current, since both positive and negative particles move together. When collisions between charged and neutral particles are important, this drift gives rise to an electric current, since the ion-neutral collision frequency is greater than the electron-neutral collision frequency, causing the ions to move slower than the electrons. This current is normal to both **E** and **B**, and is in the direction opposite to $\mathbf{v}_E$. It is known as the *Hall current*.

5.2 Solution in Cartesian Coordinates

Let us choose a Cartesian coordinate system with the *z* axis pointing in the direction of **B**, so that

$$\mathbf{B} = B\hat{\mathbf{z}} \tag{5.16}$$

$$\mathbf{E} = E_x\widehat{\mathbf{x}} + E_y\widehat{\mathbf{y}} + E_z\widehat{\mathbf{z}} \tag{5.17}$$

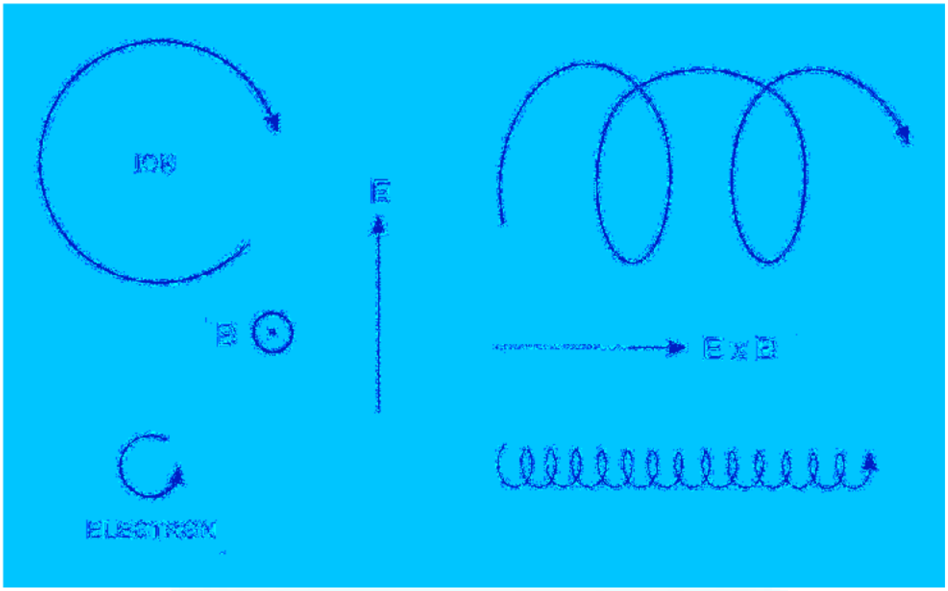

Fig. 10 Cycloidal trajectories described by ions and electrons in crossed electric and magnetic fields. The electric field **E** acting together with the magnetic flux density **B** gives rise to a drift velocity in the direction given by **E** × **B**.

Using (4.15), the equation of motion (5.1) can be written as

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}[(E_x + v_y B)\widehat{\mathbf{x}} + (E_y - v_x B)\widehat{\mathbf{y}} + E_z\widehat{\mathbf{z}}] \qquad (5.18)$$

As before, we consider, in what follows, a *positive* charge. The results for a negative charge can be obtained by changing the sign of Ω_c in the results for the positive charge.

The z component of (5.18) can be integrated directly and gives the same results expressed in (5.6) and (5.7). For the x and y components, we first take the derivative of dv_x/dt with respect to time and substitute the expression for dv_y/dt, which gives

$$\frac{d^2 v_x}{dt^2} + \Omega_c^2 v_x = \Omega_c^2 \frac{E_y}{B} \qquad (5.19)$$

This is the *inhomogeneous* differential equation for a *harmonic oscillator* of frequency Ω_c. Its solution is given by the sum of the homogeneous equation solution, given in (4.21), with a particular solution (which is clearly given by E_y/B). Thus,

$$v_x(t) = v'_\perp \sin(\Omega_c t + \theta_o) + \frac{E_y}{B} \tag{5.20}$$

where $v'_\perp$ and θ_o are integration constants. The solution for $v_y(t)$ can be obtained by substituting (5.20) directly into (5.18). Hence,

$$v_y(t) = \frac{1}{\Omega_c}\frac{dv_x}{dt} - \frac{E_x}{B} = v'_\perp \cos(\Omega_c t + \theta_o) - \frac{E_x}{B} \tag{5.21}$$

Therefore, the velocity components $v_x(t)$ and $v_y(t)$, in the plane perpendicular to **B**, oscillate at the cyclotron frequency Ω_c and with amplitude $v'_\perp$. This motion is superposed to a constant drift velocity $\mathbf{v}_E$ given by

$$\mathbf{v}^E = \frac{E_y}{B}\hat{\mathbf{x}} - \frac{E_x}{B}\hat{\mathbf{y}} \tag{5.22}$$

This expression corresponds to (5.11) when $\mathbf{B} = B\hat{\mathbf{z}}$.

One more integration of (5.20) and (5.21) gives the particle trajectory in the (x,y) plane

$$x(t) = -\frac{v'_\perp}{\Omega_c}\cos(\Omega_c t + \theta_o) + \frac{E_y}{B}t + X_o \tag{5.23}$$

$$y(t) = \frac{v'_\perp}{\Omega_c}\sin(\Omega_c t + \theta_o) - \frac{E_x}{B}t + Y_o \tag{5.24}$$

where X_o and Y_o are defined according to (4.27) and (4.28), but with $v_\perp$ replaced by $v'_\perp$.

In summary, the motion of a charged particle in uniform electrostatic and magnetostatic fields consists of three components:

(a) A constant acceleration $q\mathbf{E}_\parallel/m$ along the **B** field. If **E** = 0, the particle moves along **B** with its initial velocity.

(b) A rotation in the plane normal to **B** at the cyclotron frequency $\Omega_c = |q|\, B/m$ and radius $r_c = v'_\perp/\Omega_c$.

(c) An electromagnetic drift velocity $\mathbf{v}_E = (\mathbf{E}\times\mathbf{B})/B^2$, perpendicular to both **B** and **E**.

6. DRIFT DUE TO AN EXTERNAL FORCE

If some additional force **F** (gravitational force or inertial force, if the motion is considered in a noninertial system, for example) is present, the equation of motion (1.5) must be modified to include this force,

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F} \qquad (6.1)$$

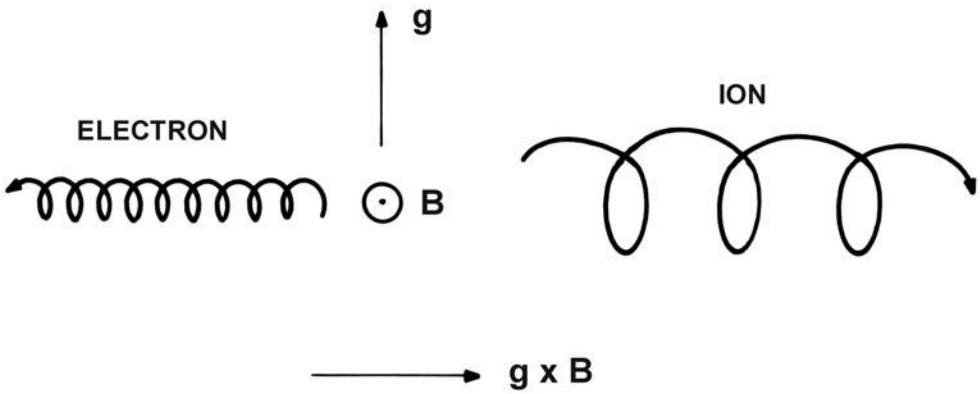

Fig.11 Drift of a gyrating particle in crossed gravitational and magnetic fields.

The effect of this force is, in a formal sense, analogous to the effect of the electric field. We assume here that **F** is uniform and constant. In analogy with the electromagnetic drift velocity $\mathbf{v}_E$, given in (5.15), the drift produced by the force **F** having a component normal to **B** is given by

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \qquad (6.2)$$

In the case of a uniform gravitational field, for example, we have $\mathbf{F} = m\mathbf{g}$, where **g** is the acceleration due to gravity, and the drift velocity is given by

$$\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g} \times \mathbf{B}}{B^2} \qquad (6.3)$$

This drift velocity depends on the ratio *m/q* and therefore it is in opposite directions for particles of opposite charge (Fig. 11). We have seen that in a coordinate system moving with the velocity $\mathbf{v}_E$, the electric field component $\mathbf{E}_\perp$ is transformed away leaving the magnetic field unchanged. The gravitational field however cannot, in this context, be transformed away.

In a collisionless plasma, associated with the gravitational drift velocity there is an electric current density, $\mathbf{J}_g$, in the direction of $\mathbf{g} \times \mathbf{B}$, which can be expressed as

$$\mathbf{J}^g = \frac{1}{\delta V}\sum_i q_i \mathbf{v}_{gi} \tag{6.4}$$

where the summation is over all charged particles contained in a suitably chosen small volume element δV. Using (6.3) we obtain

$$\mathbf{J}_g = \frac{1}{\delta V}\left(\sum_i m_i\right)\frac{\mathbf{g}\times\mathbf{B}}{B^2} = \rho_m \frac{\mathbf{g}\times\mathbf{B}}{B^2} \tag{6.5}$$

where ρ_m denotes the total mass density of the charged particles.

A comment on the validity of (6.2) is appropriate here. Since we have used the nonrelativistic equation of motion, there is a limitation on the magnitude of the force **F** in order that (6.2) be applicable. The magnitude of the transverse drift velocity is given by

$$v_D = \frac{F_\perp}{qB} \tag{6.6}$$

Hence, for the nonrelativistic equation of motion to be applicable we must have

$$\frac{F_\perp}{qB} \ll c \tag{6.7}$$

or, if **F** is due to an electrostatic field **E**,

$$\frac{E_\perp}{B} \ll c \tag{6.8}$$

For a magnetic field of 1 tesla, for example, (6.2) may be used as long as $E_\perp$ is much less than 10^8 volts/m. If these conditions are not satisfied, the problem must be treated using the relativistic equation of motion. Although the relativistic equation of motion can be integrated exactly for constant **B**, **E** and **F**, we shall not analyze this problem here. It is left as an exercise for the reader.

PROBLEMS

2.1 Calculate the cyclotron frequency and the cyclotron radius for: (a) An electron in the Earth's ionosphere at 300 km altitude, where the magnetic flux density $B \simeq 0.5 \times 10^{-4}$ tesla, considering that the electron moves at the thermal velocity (kT/m), with T = 1000 K, where k is Boltzmann's constant.
(b) A 50 MeV proton in the Earth's inner Van Allen radiation belt at

about 1.5 R_E (where R_E = 6370 km is the Earth's radius) from the center of the Earth in the equatorial plane, considering $B \simeq 10^{-5}$ tesla. (c) A 1 MeV electron in the Earth's outer Van Allen radiation belt at about 4 R_E from the center of the Earth in the equatorial plane, where $B \simeq 10^{-7}$ tesla.
(d) A proton in the solar wind with a streaming velocity of 100 km/s, ina magnetic flux density $B \simeq 10^{-9}$ tesla.
(e) A 1 MeV proton in a sunspot region of the solar photosphere, considering $B \simeq 0.1$ tesla.

2.2 For an electron and an oxygen ion O^+ in the Earth's ionosphere, at 300 km altitude in the equatorial plane, where $B \simeq 0.5 \times 10^{-4}$ tesla, calculate:
(a) The gravitational drift velocity $\mathbf{v}_g$.
(b) the gravitational current density $\mathbf{J}_g$, considering $n_e = n_i = 10^{12}$ m^{-3}. Assume that **g** is perpendicular to **B**.

2.3 Consider a particle of mass m and charge q moving in the presence of constant and uniform electromagnetic fields given by **E** $= E_o\widehat{\mathbf{y}}$ and **B** $= B_o\widehat{\mathbf{z}}$. Assuming that initially (t = 0) the particle is at rest at the origin of a Cartesian coordinate system, show that it moves on the *cycloid*

$$x(t) = \frac{E_o}{B_o}\left[t - \frac{1}{\Omega_c}\sin(\Omega_c t)\right]$$

$$y(t) = \frac{E_o}{B_o}\frac{1}{\Omega_c}[1 - \cos(\Omega_c t)]$$

Plot the trajectory of the particle in the z = 0 plane for $q > 0$ and for $q < 0$, and consider the cases when $v_\perp > v_E$, $v_\perp = v_E$ and $v_\perp < v_E$, where $v_\perp$ denotes the particle cyclotron motion velocity and v_E is the electromagnetic drift velocity.

2.4 In general the trajectory of a charged particle in crossed electric and magnetic fields is a cycloid. Show that, if $\mathbf{v} = v_o\hat{\mathbf{x}}$, $\mathbf{B} = B_o\hat{\mathbf{z}}$ and $\mathbf{E} = E_o\hat{\mathbf{y}}$, then for $v_o = E_o/B_o$ the path is a straight line. Explain how this situation can be exploited to design a mass spectrometer.

2.5 Derive the relativistic equation of motion in the form (1.4), starting from (1.1) and the relation (1.2).

2.6 Write down, in vector form, the relativistic equation of motion for a charged particle in the presence of a uniform magnetostatic field $\mathbf{B} = B_o\hat{\mathbf{z}}$, and show that its Cartesian components are given by

$$\frac{d}{dt}(\gamma v_x) = \left(\frac{qB_o}{m}\right)v_y$$

$$\frac{d}{dt}(\gamma v_y) = -\left(\frac{qB_o}{m}\right)v_x$$

$$\frac{d}{dt}(\gamma v_z) = 0$$

where

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}}$$

and where $\beta = v/c$. Show that the velocity and trajectory of the charged particle are given by the same formulas as in the nonrelativistic case, but with Ω_c replaced by $|q|\, B_o/(m\gamma)$.

2.7 Analyze the motion of a relativistic charged particle in the presence of crossed electric (**E**) and magnetic (**B**) fields which are constant in time and uniform in space. What coordinate transformation must be made in order to transform away the transversal electric field? Derive equations for the velocity and trajectory of the charged particle.