POISSON'S AND LAPLACE'S EQUATIONS

A study of the previous chapter shows that several of the analogies used to obtain experimental field maps involved demonstrating that the analogous quantity satisfies Laplace's equation. This is true for small deflections of an elastic membrane, and we might have proved the current analogy by showing that the direct-current density in a conducting medium also satisfies Laplace's equation. It appears that this is a fundamental equation in more than one field of science, and, perhaps without knowing it, we have spent the last chapter obtaining solutions for Laplace's equation by experimental, graphical, and numerical methods. Now we are ready to obtain this equation formally and discuss several methods by which it may be solved analytically.

It may seem that this material properly belongs before that of the previous chapter; as long as we are solving one equation by so many methods, would it not be fitting to see the equation first? The disadvantage of this more logical order lies in the fact that solving Laplace's equation is an exercise in mathematics, and unless we have the physical problem well in mind, we may easily miss the physical significance of what we are doing. A rough curvilinear map can tell us much about a field and then may be used later to check our mathematical solutions for gross errors or to indicate certain peculiar regions in the field which require special treatment.

With this explanation let us finally obtain the equations of Laplace and Poisson.

7.1 POISSON'S AND LAPLACE'S EQUATIONS

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_v \qquad (l)$$

the definition of D,

$$D=eE \qquad (2)$$

and the gradient relationship,

$$\mathbf{E} = -\nabla V \qquad (3)$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

or

$$v.vv=—\_ \qquad (4)$$

for a homogeneous region in which € is constant.

Equation (4) is Poisson's equation, but the "double V" operation must be interpreted and expanded, at least in cartesian coordinates, before the equation can be useful. In cartesian coordinates,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\quad} + \frac{\partial A_y}{\quad} + \underline{OA:}$$

ax ay a:

$$\nabla V = \frac{\partial V}{\quad}\mathbf{a}_x + \frac{\partial V}{\quad}\mathbf{a}_y + \frac{\partial V}{\quad}\mathbf{a}_z$$

Ox ^{x} ay ^{Y} az

and therefore

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

02 v 02 v 02 v

(5) ⁻ Ox2 ay2 ∂z^2

Usually the operation V • V is abbreviated 17² (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in (5), and we have

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad (6)$$

in cartesian coordinates.

If pt, = O, indicating zero volume charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

v2v=o (7)

which is Laplace's equation. The ∇^2 operation is called the Laplacian of V.

In cartesian coordinates Laplace's equation is

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(cartesian)} \qquad (8)$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\text{v2v=} \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad \text{(cylindrical)} \qquad (9)$$

and in spherical coordinates is

v2v=			sin 0— +		2	(spherical)	(10)
	r^2 Or	Or	r^2 sin 9 00	00	r2 $\sin^2$ Ø 0+		

These equations may be expanded by taking the indicated partial derivatives, but it is usually more helpful to have them in the forms given above; furthermore, it is much easier to expand them later if necessary than it is to put the broken pieces back together again.

Laplace's equation is all-embracing, for, applying as it does wherever volume charge density is zero, it states that every conceivable configuration of electrodes or conductors produces a field for which V2 V = O. All these fields are different, with different potential values and different spatial rates of change, yet for each of them V^2 V = O. Since every field (if m, = O) satisfies Laplace's equation, how c»an we expect to reverse the procedure and use Laplace's equation to find one specific field in which we happen to have an interest? Obviously, more information is required, and we shall find that we must solve Laplace's equation subject to certain boundary conditions.

Every physical problem must contain at least one conducting boundary and usually contains two or more. The potentials on these boundaries are assigned values, perhaps Vo, VI, or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved in this chapter. In other types of problems, the boundary conditions take the form of specified values of E on an enclosing surface, or a mixture of known values of V and E.

Before using Laplace's equation or Poisson's equation in several examples, we must pause to show that if our answer satisfies Laplace's equation and also satisfies the boundary conditions, then it is the only possible answer. It would be very distressing to work a problem by solving Laplace's equation with two different approved methods and then to obtain two different answers. We shall show that the two answers must be identical.

D7.1. Calculate numerical values for V and at point P in free space if: 4yz

(a) $V = \frac{\cdots}{x^2+1}$, a at PO. 2.3); (b) V=5p2cos2+, at $P(\rho = 3, \phi = \frac{\pi}{3},$ z-=2); (c) $V = \frac{2\cos\phi}{r^2}$, at $P(r$

0 0

= 0.5, 0=45, 60).

Ans. 12V, —106.2pC/m^3; 22.5V, O; 4V, —141.7pC/rn^3

7.2 UNIQUENESS THEOREM

Let us assume that we have two solutions of Laplace's equation, VI and V2, both general functions of the coordinates used. Therefore v^2 Vl = 0

and v2V2=o

from which v^2(Vl — V2) = o

Each solution must also satisfy the boundary conditions, and if we represent the given potential values on the boundaries by Vb, then the value of VI on the boundary Vib and the value of V2 on the boundary V2b must both be identical to VB,

Vib$= V_{2b}$

Vib$- V_{2b} =$

In Sec. 4.8, Eq. (44), we made use of a vector identity,

V. (VI)) = • D) +1).(∇V)

which holds for any scalar V and any vector D. For the present application we shall select VI — V2 as the scalar and V(VI — V2) as the vector, giving

$$\nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] \equiv (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)]$$
$$+ \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2)$$

which we shall integrate throughout the volume enclosed by the boundary surfaces specified:

$$\int_{vol} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv$$

[V(VI $\equiv \int_{vol} (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)]dv + \int$ — V2)J^2 dv (1 1)

vol

The divergence theorem allows us to replace the volume integral on the left side of the equation by the closed surface integral over the surface surrounding the volume. This surface consists of the boundaries already specified on which Vib = V2b, and therefore

$$\int_{vol} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv = \oint_S [(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})] \cdot d\mathbf{S} = 0$$

One of the factors of the first integral on the right side of (I I) is V • V(VI — V2), or V^2(Vl — V2), which is zero by hypothesis, and therefore that integral is zero. Hence the remaining volume integral must be zero:

$$\int_{vol} [\nabla(V_1 - V_2)]^2\, dv = 0$$

There are two reasons why an integral may be zero: either the integrand (the quantity under the integral sign) is everywhere zero, or the integrand is positive in some regions and negative in others, and the contributions cancel algebraically. In this case the first reason must hold because [V(VI — cannot be negative. Therefore

[V(VI — = 0

and

V(VI - J'2) = O

Finally, if the gradient of VI — is everywhere zero, then VI — V2 cannot change with any coordinates and

VI — V2 = constant

If we can show that this constant is zero, we shall have accomplished our proof. The constant is easily evaluated by considering a point on the boundary. Here VI — V2 = Vib — V2h = 0, and we see that the constant is indeed zero, and there-

fore

giving two identical solutions.

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. The uniqueness theorem also applies to Poisson's equation, for if V2Vl = — p,,/e and V^2 V2 = —pv/e, then V^2 (Vl — V2) = O as before. Boundary conditions still require that Vib — 172b = 0, and the proof is identical from this point.

This constitutes the proof of the uniqueness theorem. Viewed as the answer to a question, "How do two solutions of Laplace's or Poisson's equation compare if they both satisfy the same boundary conditions?" the uniqueness theorem should please us by its ensurance that the answers are identical. Once we can find any method of solving Laplace's or Poisson's equation subject to given boundary conditions, we have solved our problem once and for all. No other method can ever give a different answer.

1)7.2. Consider the two potential fields VI = y and V2 = y siny. (a) Is V2 VI = O? (b) Is Is V, Is (e)ls Is V2 = at y = r? (g) Are VI and V2 identical? (h) Why does the uniqueness theorem not apply?

Ans. Yes; yes; yes; yes; yes; yes; no; boundary conditions not given for a closed surface

7.3 EXAMPLES OF THE SOLUTION OF LAPLACE'S EQUATION

Several methods have been developed for solving the second-order partial differential equation known as Laplace's equation. The first and simplest method is that of direct integration, and we shall use this technique to work several examples in various coordinate systems in this section. In Sec. 7.5 one other method will be used on a more difficult problem. Additional methods, requiring a more advanced mathematical knowledge, are described in the references given at the end of the chapter.

The method of direct integration is applicable only to problems which are "one-dimensional," or in which the potential field is a function of only one of the three coordinates. Since we are working with only three coordinate systems, it might seem, then, that there are nine problems to be solved, but a little

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reflection will show that a field which varies only with x is fundamentally the same as a field which varies only with y. Rotating the physical problem a quarter turn is no change. Actually, there are only five problems to be solved, one in cartesian coordinates, two in cylindrical, and two in spherical. We shall enjoy life to the fullest by solving them all.

Illi*Examp1e 7.1

Let us assume that V is a function only of x and worry later about which physical problem we are solving when we have a need for boundary conditions. Laplace's equation reduces to

and the partial derivative may be replaced by an ordinary derivative, since V is not a function of y or z,

We integrate twice, obtaining

$$V = Ax+B \tag{12}$$

where A and B are constants of integration. Equation (12) contains two such constants, as we should expect for a second-order differential equation. These constants can be determined only from the boundary conditions.

What boundary conditions should we supply? They are our choice, since no physical problem has yet been specified, with the exception of the original hypothesis that the potential varied only with x. We should now attempt to visualize such a field. Most of us probably already have the answer, but it may be obtained by exact methods.

Since the field varies only with x and is not a function of y and z, then V is a constant if x is a constant or, in other words. the equipotential surfaces are described by setting x constant. These surfaces are parallel planes normal to the x axis. The field is thus that of a parallel-plate capacitor, and as soon as we specify the potential on any two planes, we may evaluate our constants of integration.

To be very general, let V = VI at x = and V' = at x = .v2. These values are then substituted into (12), giving

$$V_1 = Ax_1 + B \qquad V_2 = Ax_2 + B$$

$$A = \frac{V_1 - V_2}{x_1 - x_2} \qquad B = \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$$

$$V = \frac{V_1(x - x_2) - V_2(x - x_1)}{x_1 - x_2}$$

A simpler answer would have been obtained by choosing simpler boundary conditions. If we had fixed V = O at x = O and V = Vo at x = d, then and

$$V = \frac{V_0 x}{d} \tag{13}$$

Suppose our primary aim is to find the capacitance of a parallel-plate capacitor. We have solved Laplace's equation, obtaining (12) with the two constants A and B. Should they be evaluated or left alone? Presumably we are not interested in the potential field itself, but only in the capacitance, and we may continue successfully with A and B or we may simplify the algebra by a little foresight. Capacitance is given by the ratio of charge to potential difference, so we may choose now the potential difference as Yo, which is equivalent to one boundary condition, and then choose whatever second boundary condition seems to help the form of the equation the most. This is the essence of the second set of boundary conditions which produced (13). The potential difference was fixed as Vo by choosing the potential of one plate zero and the other Yo; the location of these plates was made as simple as possible by letting V = O al x = O.

Using (13), then, we still need the total charge on either plate before the capacitance can be found. We should remember that when we first solved this capacitor problem in Chap. 5, the sheet of charge provided our starting point. We did not have to work very hard to find the charge, for all the fields were expressed in terms of it. The work then was spent in finding potential difference. Now the problem is reversed (and simplified).

The necessary steps are these, after the choice of boundary conditions has been made:

1. Given V, use —VV to find E.
2. Use D = to find D.
3. Evaluate D at either capacitor plate, D = DS = max.
4. Recognize that ps = DN.
5. Find Q by a surface integration over the capacitor plate, Q= Js psdS.

Here we have

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

Vo

ax

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon \frac{V_0}{d}$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S$$

DAI

VOS and the capacitance —e is

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = \cdot$$

$$\boxed{C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}}$$

We shall use this procedure several times in the examples to follow.

Example 7.2

Since no new problems are solved by choosing fields which vary only with y or with in cartesian coordinates, we pass on to cylindrical coordinates for our next example. Variations with respect to : are again nothing new, and we next assume variation with respect to p only. Laplace's equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

or

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

Noting the p in the denominator, we exclude p = O from our solution and then multiply by p and integrate,

$$\rho \frac{dV}{d\rho} = A$$

dp

rearrange, and integrate again,

$$V = A \ln \rho + B \tag{15}$$

The equipotential surfaces are given by p = constant and are cylinders, and the problem is that of the coaxial capacitor or coaxial transmission line. We choose a potential difference Of vo by letting V = vo at p = a, V = O at p = b, b > a, and Obtain

In(b/p) V % In(b/a)

(16)

from which

E= vo — —I —p In(b/a)

$$D_{N(\rho=a)} = \frac{\epsilon V_0}{a \ln(b/a)}$$

a In(h/a)

$$Q = \frac{\epsilon V_0 2\pi a L}{a \ln(b/a)}$$

a In(b/a)

$C = \frac{2\pi\epsilon L}{\ln(b/a)}$ – In(b/a)

(17)

which agrees with our results in Chap. 5.

Example 7.3

Now let us assume that V is a function only of in cylindrical coordinates. We might look at the physical problem first for a change and see that equipotential surfaces are given by = constant. These are radial planes. Boundary conditions might be V = O at ϕ= 0 and V Vo at d) = a, leading to the physical problem detailed in Fig. 7.1.

Laplace's equation is now

I a2v

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

We exclude p = O and have

The solution is

$$V = A\phi + B$$

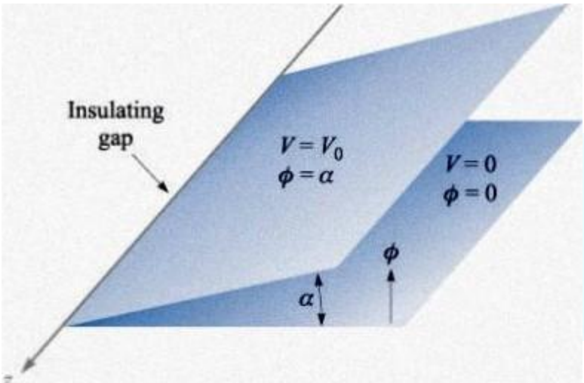

FIGURE 7.1
Two infinite radial planes with an interior angle a. An infinitesimal insulating gap exists at p = O. The potential field may be found by applying Laplace's equation in cylindrical coordinates.

The boundary conditions determine A and B, and

$$V = V_0 \frac{\tau}{\alpha_{q\}}} \tag{18}$$

Taking the gradient of (18) produces the electric field intensity,

Voa.
up (19)

and it is interesting to note that E is a function of p and not of O. This does not contradict our original assumptions, which were restrictions only on the potential field. Note, however, that the vector field E is a function of 4.

A problem involving the capacitance of these two radial planes is included at the end Of the chapter.

Example 7.4

We now turn to spherical coordinates, dispose immediately ofvariations with respect to ϕ only as having just been solved, and treat first V = V(r).

The details are left for a problem later, but the final potential field is given by

$$V = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} \tag{20}$$

where the boundary conditions are evidently V = 0 at r = and V = Vo at r = a, b > a. The problem is that of concentric spheres. The capacitance was found previously in Sec. 5.10 (by a somewhat different method) and is

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}} \tag{21}$$

Example 7.5

In spherical coordinates we now restrict the potential function to V' = V(e), Obtaining

_ _sin O =r2 sin 0

do

We exclude r O and O = O or and have

sine— = A

The second integral is then

$$V = \int \frac{A\,d\theta}{\sin\theta} + B$$

which is not as obvious as the previous ones. From integral tables (or a good memory) we have

V = A In tan— + B

The equipotential surfaces are cones. Fig. 7.2 illustrates the case where V = O at O = r/2 and V = Vo at = a, a < n/2. We obtain

$$\boxed{V = V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)}} \qquad (22)$$

In order to find the capacitance between a conducting cone with its vertex separated from a conducting plane by an infinitesimal insulating gap and its axis normal to the plane, let us first find the field strength:

$$\mathbf{E} = -\nabla V = \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\left(\tan\frac{\alpha}{2}\right)}\mathbf{a}_\theta$$

The surface charge density on the cone is then

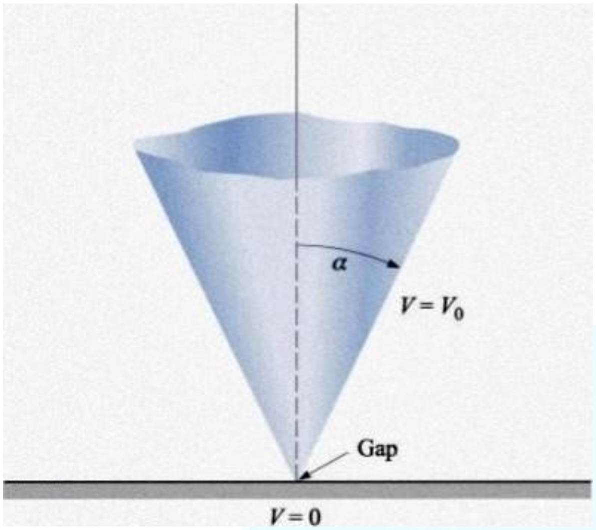

FIGURE 7.2

For the cone 9 = a at Vo and the plane 9=r/2 at V = 0, the potential field is given by V = $V_0[\ln(\tan\theta/2)]/[\ln(\tan\alpha/2)]$.

Ps = $\frac{-\epsilon V_0}{r \sin\alpha \ln\left(\tan\frac{\alpha}{2}\right)}$

producing a total charge Q,

$$Q = \frac{-\epsilon V_0}{\sin\alpha \ln\left(\tan\frac{\alpha}{2}\right)} \int_0^{\infty} \int_0^{2\pi} \frac{r \sin\alpha \, d\phi \, dr}{r}$$

$$= \frac{-2\pi\epsilon_0 V_0}{\ln\left(\tan\frac{\alpha}{2}\right)} \int_0^{\infty} dr$$

This leads to an infinite value of charge and capacitance, and it becomes necessary to consider a cone of finite size. Our answer will now be only an approximation, because the theoretical equipotential surface is 0 = a, a conical surface extending from r = 0 to r = 00, whereas our physical conical surface extends only from r = 0 to, say, r = The approximate capacitance is

$$C \doteq \frac{\quad}{\ln\left(\cot\frac{\alpha}{2}\right)} \qquad (23)$$

2rrerl

If we desire a more accurate answer, we may make an estimate of the capacitance of the base of the cone to the zero-potential plane and add this amount to our answer above. Fringing, or nonuniform, fields in this region have been neglected and introduce an additional source of error.

1)7.3. Find IF-I at P(3, l, 2) for the field of: (a) two coaxial conducting cylinders, V = 50 V at p = 2m, and V = 20 V at p = 3m; (b) two radial conducting planes, v = 50V at 10^0, and v = 20V at $4=30^0$.

Ans. 23.4V/m; 27.2V/m

7.4 EXAMPLE OF THE SOLUTION OF POISSON'S EQUATION

To select a reasonably simple problem which might illustrate the application of Poisson's equation, we must assume that the volume charge density is specified. This is not usually the case, however; in fact, it is often the quantity about which we are seeking further information. The type of problem which we might encounter later would begin with a knowledge only of the boundary values of the potential, the electric field intensity, and the current density. From these we would have to apply Poisson's equation, the continuity equation, and some relationship expressing the forces on the charged particles, such as the Lorentz force equation or the diffusion equation, and solve the whole system of equations simultaneously. Such an ordeal is beyond the scope of this text, and we shall therefore assume a reasonably large amount of information.

As an example, let us select a pn junction between two halves of a semiconductor bar extending in the x direction. We shall assume that the region for x < O is doped p type and that the region for x > 0 is n type. The degree of doping is identical on each side of the junction. To review qualitatively some of the facts about the semiconductor junction, we note that initially there are excess holes to the left of the junction and excess electrons to the right. Each diffuses across the junction until an electric field is built up in such a direction that the diffusion current drops to zero. Thus, to prevent more holes from moving to the right, the electric field in the neighborhood of the junction must be directed to the left; Ex is negative there. This field must be produced by a net positive charge to the right of the junction and a net negative charge to the left. Note that the layer of positive charge consists of two parts—the holes which have crossed the junction and the positive donor ions from which the electrons have departed. The

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negative layer of charge is constituted in the opposite manner by electrons and negative acceptor ions.

The type of charge distribution which results is shown in Fig. 7.3a, and the negative field which it produces is shown in Fig. 7.3b. After looking at these two figures, one might profitably read the previous paragraph again.

A charge distribution of this form may be approximated by many different expressions. One of the simpler expressions IS

m, = 2ptÅ) sech — tanh — (24)

which has a maximum charge density pt,,max = pto that occurs at x = 0.88 la. The maximum charge density is related to the acceptor and donor concentrations Na and Nd by noting that all the donor and acceptor ions in this region (the depletion layer) have been stripped of an electron or a hole, and thus

$$\rho_{v0} = eN_a = eN_d$$

Let us now solve Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

subject to the charge distribution assumed above,

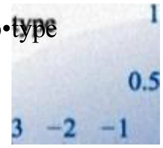
(1² v2 ρ
. — ——•sech tanh—
dx2 a a

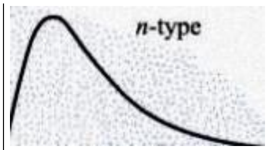
in this one-dimensional problem in which variations with y and z are not present. We integrate once,

$$\frac{dV}{} = -\frac{2\rho_{v0}a}{\epsilon} \text{sech} \frac{x}{} dx$$

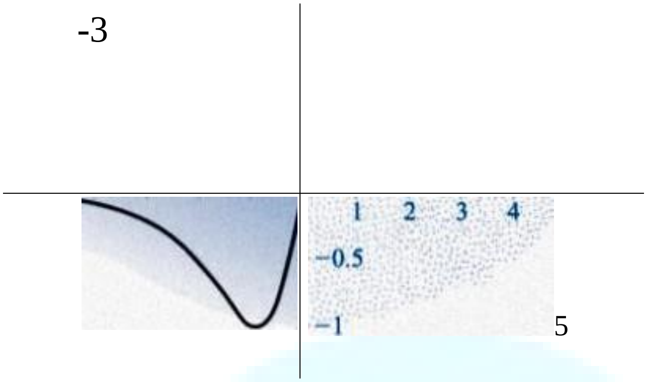
p•type

-4



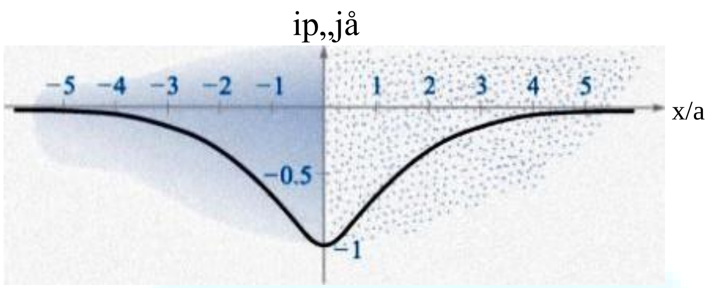


and field



obtain the electric intensity,

(a)



(b)

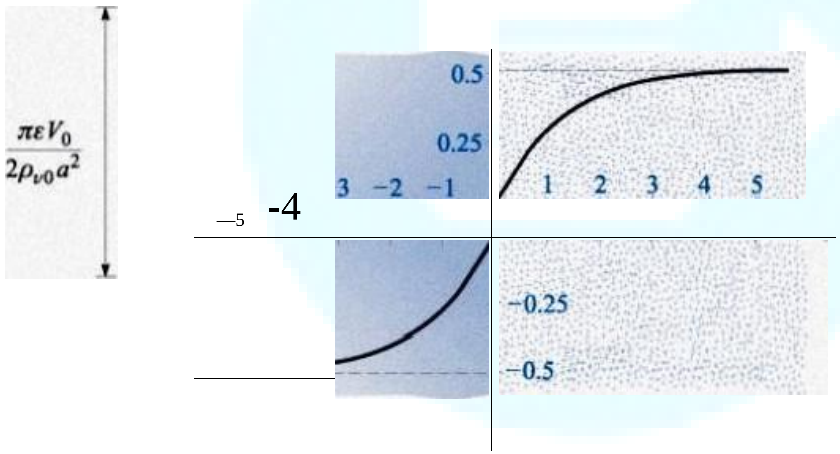


FIGURE 7.3

(a) The charge density. (b) the electric field intensity, and (c) the potential are plotted for a pn junction as functions of distance from the center of the junction. The p-type material is on the left. and the n-type is on the right.

2ptoa x
_ ____sech — Cl

To evaluate the constant of integration G, we note that no net charge density and no fields can exist far from the junction. Thus, as x —+ +00, Ex must approach zero. Therefore Cl = O, and

2p,Åja____x
_ sech_ (25)

Integrating again,

=4ptoa2tan¯l e^{x}/a + C'2

Let us arbitrarily select our zero reference of potential at the center of the junction, x O,

4ptoa2 π
_ C2

and finally,

4ptoa2 tan—I ex/a _4) (26)
ϵ

Fig. 7.3 shows the charge distribution (a), electric field intensity (b), and the potential (c), as given by (24), (25), and (26), respectively.

The potential IS constant once we are a distance of about 4a or 5a from the junction. The total potential difference Vo across the junction is obtained from
(26),

2rrpzoa2
Vo = vx.N$- V_{x\to -\infty} =$ (27)

This expression suggests the possibility of determining the total charge on one side of the junction and then using (27) to find a junction capacitance. The total positive charge is

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$$Q = S\int_0^{\infty} 2\rho_{v0}\,\mathrm{sech}\frac{x}{a}\tanh\frac{x}{a}\,dx = 2\rho_{v0}aS$$

where S is the area of the junction cross section. If we make use of (27) to eliminate the distance parameter a, the charge becomes

$$Q=s\sqrt{\frac{2\rho_{v0}\epsilon V_0}{\pi}} \qquad (28)$$

Since the total charge is a function of the potential difference, we have to be careful in defining a capacitance. Thinking in "circuit" terms for a moment,

$$I = \frac{dQ}{dt} = C\frac{dV_0}{dt}$$

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and thus dQ

$$C = \frac{dQ}{dV_0}$$

By differentiating (28) we therefore have the capacitance,

$$C = \sqrt{\frac{\rho_{v0}\epsilon}{2\pi V_0}}S = \frac{\epsilon S}{2\pi a} \qquad (29)$$

The first form of (29) shows that the capacitance varies inversely as the square root of the voltage. That is, a higher voltage causes a greater separation of the charge layers and a smaller capacitance. The second form is interesting in that it indicates that we may think of the junction as a parallel-plate capacitor with a "plate" separation of 2na. In view of the dimensions of the region in which the charge is concentrated, this is a logical result.

Poisson's equation enters into any problem involving volume charge density. Besides semiconductor diode and transistor models, we find that vacuum tubes, magnetohydrodynamic energy conversion, and ion propulsion require its use in constructing satisfactory theories.

D7.4. In the neighborhood of a certain semiconductor junction the volume charge density is given by pe = 750 sech tanh rrxC/m3. The dielectric constant of the semiconductor material is 10 and the junction area is 2 x 10¯7 m2. Find: (a) Vo; (h) C; (c) E at the junction.

Ans. 2.70V; 8.85pF; 2.70MV/m

D7.5. Given the volume charge density pa, = —2 x 107eoVRC/m3 in free space, let V = 0 at x = O and V = 2V at x = 2.5mm. At x = I mm, find: (a) V; (b) Ex.

Ans. 0.302 V; —555 V/m

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