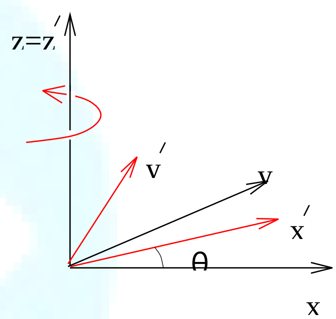
6. Non-Inertial Frames

We stated, long ago, that inertial frames provide the setting for Newtonian mechanics. But what if you, one day, find yourself in a frame that is not inertial? For example, suppose that every 24 hours you happen to spin around an axis which is 2500 miles away. What would you feel? Or what if every year you spin around an axis 36 million miles away? Would that have any e↵ect on your everyday life?

In this section we will discuss what Newton's equations of motion look like in noninertial frames. Just as there are many ways that an animal can be not a dog, so there are many ways in which a reference frame can be non-inertial. Here we will just consider one type: reference frames that rotate. We'll start with some basic concepts.

6.1 Rotating Frames

Let's start with the inertial frame *S* drawn in the figure with coordinate axes *x*, *y* and *z*. Our goal is to understand the motion of particles as seen in a non-inertial frame S_0, with axes x_0, y_0 and z_0, which is rotating with respect to *S*. We'll denote the angle between the *x*-axis of *S* and the $x_0$axis of S_0 as ✓. Since S_0 is rotating, we clearly have ✓= ✓(*t*) and ✓˙ = 0.6



Our first task is to find a way to describe the rotation of

Figure 31: the

axes. For this, we can use the angular velocity vector *!* that we introduced in the last section to describe the motion of particles. Consider a particle that is sitting stationary in the S_0 frame. Then, from the perspective of frame *S* it will appear to be moving with velocity

$$\dot{\mathbf{r}} = ! \rightarrow \mathbf{r}$$

where, in the present case, *!* = ✓˙**z**ˆ. Recall that in general, |*!*| = ✓˙ is the angular speed, while the direction of *!* is the axis of rotation, defined in a right-handed sense.

We can extend this description of the rotation of the axes of S_0 themselves. Let $\mathbf{e}_{i0}$, *i* = 1,2,3 be the unit vectors that point along the x_0, y_0 and z_0 directions of S_0. Then these also rotate with velocity

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i$$

This will be the main formula that will allow us to understand motion in rotating frames.

6.1.1 Velocity and Acceleration in a Rotating Frame

Consider now a particle which is no longer stuck in the S_0 frame, but moves on some trajectory. We can measure the position of the particle in the inertial frame S, where, using the summation convention, we write

$$\mathbf{r} = r_i \mathbf{e}_i$$

Here the unit vectors $\mathbf{e}_i$, with $i = 1,2,3$ point along the axes of S. Alternatively, we can measure the position of the particle in frame S_0, where the position is

$$\mathbf{r} = r'_i \mathbf{e}'_i$$

Note that the position vector $\mathbf{r}$ is the same in both of these expressions: but the coordinates r_i and r_{i0} di↲er because they are measured with respect to di↲erent axes. Now, we can compute an expression for the velocity of the particle. In frame S, it is simply

$$\dot{\mathbf{r}} = \dot{r}_i \mathbf{e}_i \tag{6.1}$$

because the axes $\mathbf{e}_i$ do not change with time. However, in the rotating frame S_0, the velocity of the particle is

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}_{i0} \mathbf{e}_{i0} + r_{i0} \dot{\mathbf{e}}_{i0} \\ &= \dot{r}_{i0} \mathbf{e}_{i0} + r_{i0} ! \rightarrow \mathbf{e}_{i0} \\ &= \dot{r}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{r} \end{aligned} \tag{6.2}$$

We'll introduce a slightly novel notation to help highlight the physics hiding in these two equations. We write the velocity of the particle as seen by an observer in frame S as

$$\left(\frac{d\mathbf{r}}{dt}\right)_S = \dot{r}_i \mathbf{e}_i$$

Similarly, the velocity as seen by an observer in frame S_0 is just

$$\left(\frac{d\mathbf{r}}{dt}\right)_{S'} = \dot{r}'_i \mathbf{e}'_i$$

From equations (6.1) and (6.2), we see that the two observers measure di↲erent velocities,

$$\left(\frac{d\mathbf{r}}{dt}\right)_S = \left(\frac{d\mathbf{r}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r} \qquad (6.3)$$

This is not completely surprising: the di↵erence is just the relative velocity of the two frames.

What about acceleration? We can play the same game. In frame S, we have

$$\ddot{\mathbf{r}} = \ddot{r}_i\mathbf{e}_i$$

while in frame S_0, the expression is a little more complicated. Di↵erentiating (6.2) once more, we have

$$\ddot{\mathbf{r}} = \ddot{r}_{i0}\mathbf{e}_{i0} + \dot{r}_{i0}\dot{\mathbf{e}}_{i\,0} + \dot{r}_{i0}! \to \mathbf{e}_{i0} + r_{i0}\dot{!} \to \mathbf{e}_{i0} + r_{i0}! \to \dot{\mathbf{e}}_{i\,0}$$

$$= \ddot{r}_{i0}\mathbf{e}_{i0} + 2\dot{r}_i{}^0! \to \mathbf{e}_i{}^0 + \dot{!} \to \mathbf{r} + r_i{}^0! \to (! \to \mathbf{e}_i{}^0)$$

As with velocities, the acceleration seen by the observer in S is $\ddot{r}_i\mathbf{e}_i$ while the acceleration seen by the observer in S' is $\ddot{r}'_i\mathbf{e}'_i$. Equating the two equations above gives us

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_S = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (6.4)$$

This equation contains the key to understanding the motion of particles in a rotating frame.

6.2 Newton's Equation of Motion in a Rotating Frame

With the hard work behind us, let's see how a person sitting in the rotating frame S_0 would see Newton's law of motion. We know that in the inertial frame S, we have

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_S = \mathbf{F}$$

So, using (6.4), in frame S_0, we have

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} = \mathbf{F} \quad 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} \quad m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (6.5)$$

In other words, to explain the motion of a particle an observer in S_0 must invoke the existence of three further terms on the right-hand side of Newton's equation. These are called *fictitious forces*. Viewed from S_0, a free particle doesn't travel in a straight line and these fictitious forces are necessary to explain this departure from uniform motion. In the rest of this section, we will see several examples of this.

The $2m\boldsymbol{\omega}\times\dot{\mathbf{r}}$ term in (6.5) is the *Coriolis force*; the $m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ term is called the *centrifugal force*; the $m\dot{\boldsymbol{\omega}}\times\mathbf{r}$ term is called the *Euler force*.

The most familiar non-inertial frame is the room you are sitting in. It rotates once per day around the north-south axis of the Earth. It further rotates once a year about the Sun which, in turn, rotates about the centre of the galaxy. From these time scales, we can easily compute $\omega = |\boldsymbol{\omega}|$.

The radius of the Earth is $R_{\text{Earth}} \approx 6 \times 10^3\ km$. The Earth rotates with angular frequency

$$\omega_{\text{rot}} = \frac{2\pi}{1\ \text{day}} \approx 7 \times 10^{-5}\ s^{-1}$$

The distance from the Earth to the Sun is $a_e \approx 2\times 10^8\ km$. The angular frequency of the orbit is

$$\omega_{\text{orb}} = \frac{2\pi}{1\ \text{year}} \approx 2 \times 10^{-7}\ s^{-1}$$

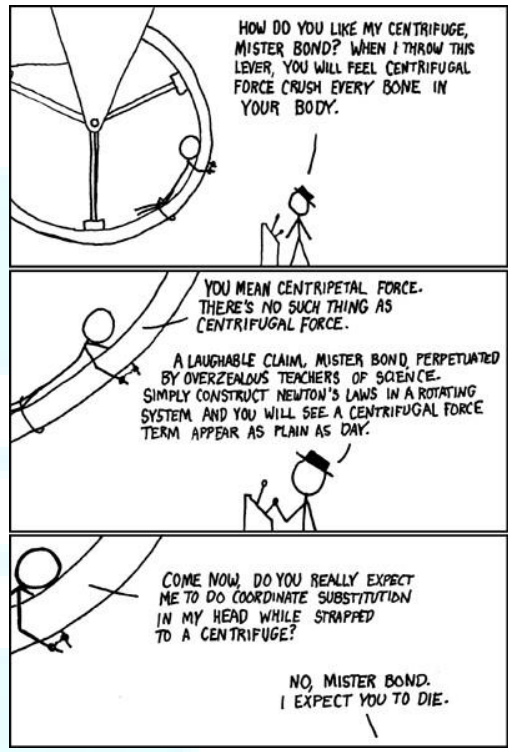


Figure 32: xkcd.com

It should come as no surprise to learn that $\omega_{\text{rot}}/\omega_{\text{orb}} = T_{\text{orb}}/T_{\text{rot}} \approx 365$.

In what follows, we will see the effect of the centrifugal and Coriolis forces on our daily lives. We will not discuss the Euler force, which arises only when the speed of the rotation changes with time. Although this plays a role in various funfair rides, it's not important in the frame of the Earth. (The angular velocity of the Earth's rotation does, in fact, have a small, but non-vanishing, $\dot{\omega}$ due to the precession and nutation of the

Earth's rotational axis. However, it is tiny, with $\dot{!}$ ☒ $!^2$ and, as far as I know, the resulting Euler force has no consequence).

Inertial vs Gravitational Mass Revisited

Notice that all the fictitious forces are proportional to the inertial mass *m*. There is no mystery here: it's because they all originated from the "ma" side of "F=ma" rather than "F" side. But, as we mentioned in Section 2, experimentally the gravitational force also appears to be proportional to the inertial mass. Is this evidence that gravity too is a fictitious force? In fact it is. Einstein's theory of general relativity recasts gravity as the fictitious force that we experience due to the curvature of space and time.