ENTRI

1.4 Phase Space

If the trajectory of the system in configuration space, F(t), is known, the velocity as a function of time, F(t) is also determined. As the mass of the particle is simply a physical constant, the momentum F = rnF contains the same information as the velocity. Viewed as functions Of time, this gives nothing beyond the information in the trajectory. But at any given time, F and F provide a complete set of initial conditions, while F alone does not. We define phase space as the set of possible positions and momenta for the system at some instant. Equivalently, it is the set of possible initial conditions, or the set of possible motions obeying the equations of motionfl For a single particle in cartesian coordinates, the six coordinates of phase

This space is called the tangent bundle to configuration space. For cartasian coordinates it is almost identical to phase space, which is in general the "cotangent bundle" to configuration space.

9 As each initial condition gives rise to a unique future development of a trajectory, there is an isomorphism between initial conditions and allowed trajectories.

space are the three components of F and the three components of F. At any instant of time, the system is represented by a point in this space, called the phase point, and that point moves with time according to the physical laws of the system. These laws are embodied in the force function, which we now consider as a function of F rather than 'F, in addition to F and t. We may write these equations as

$$\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m},$$
$$\frac{d\vec{p}}{dt} = \vec{F}(\vec{r},\vec{p},t).$$

Note that these are first order equations, which means that the motion of the point representing the system in phase space is completely determinee by where the phase point is. This is to be distinguished from the trajectory in configuration space, where in order to know the trajectory you must have not only an initial point (position) but also its initial time derivative.

1.4.1 Dynamical Systems

We have spoken of the coordinates of phase space for a single particle as rand F, but from a mathematical point of view these together give the coordinates of the phase point in phase space. We might describe these coordinates in terms of a six dimensional vector = (m, 7%2, r•3, PI, P2, P3). The physical laws determine at each point a velocity function for the phase point as it moves through phase space, dri= V (77, t), (1.13)

which gives the velocity at which the phase point representing the system moves through phase space. Only half of this velocity is the ordinary velocity, while the other half represents the rapidity with which the momentum is changing, i.e. the force. The path traced by the phase point as it travels through phase space is called the phase curve.

For a system of n particles in three dimensions, the complete set of initial conditions requires 3n spatial coordinates and 3n momenta, so phase space is 6n dimensional. While this certainly makes visualization diffcult, the large

10 We will assume throughout that the force function is a well defined continuous function of its arguments.

dimensionality is no hindrance for formal developments. Also, it is sometimes possible to focus on particular dimensions, or to make generalizations of ideas familiar in two and three dimensions. For example, in discussing integrable systems O), we will find that the rnotion of the phase point is confined to a 3n-dirnensional torus, a generalization Of one and two dimensional tori, which are circles and the surface of a donut respectively.

Thus for a system composed of a finite lltunbcr of particles, the dynamics is determined by the first order ordinary differential equation (ED, formally a very simple equation. All of the complication of the physical situation is hidden in the large dimensionality Of the dependent variable and in the functional dependence of the velocity function V (i, t) on it.

There are other systems besides Newtonian mechanics which are controlled by equation (LJJ, with a suitable velocity function. Collectively these are known as dynamical systems. For example, individuals of an asexual mutually hostile species might have a fixed birth rate b and a death rate proportional to the population, so the population would obey the logistic equatiore.] dp/dt == bp— cp2, a dynamical system with a one-dimensional space for its dependent variable. T11C populations of three competing species could be described by eq. (D with in three dimensions.

The dimensionality d of Fin C) is called the order of the dynarnical system. A d'th order differential equation ill one independent variable may always be recast as a first order differential equation in d variables, so it is one example of a d'th order dynarmcal system. The space of these dependent variables is called the phase space of the dynamical system. Newtonian systems always give rise to an even-order system, because each spatial coordinate is paired with a momentum. For n particles unconstrained in D dimensions, the order of the dynamical system is d = 2nD. Even for constrained Newtonian systems, there is always a pairing of coordinates and momenta, whicll gives a restricting structure, called the symplectic structure on phase space.

If the force function does not depend explicitly on time, we say the system is autonomous. The velocity function has no explicit dependance on time, $\vec{V}$ = 17 (j, and is a time-independent vector field on pllase space, which we can indicate by arrows just as we might the electric field in ordinary space, or the velocity field of a fluid in motion. This gives a visual indication of

[1] This is not to be confused with the simpler logistic map, which is a recursion relation with the same form but with solutions displayin a very different behavior.

12This will be discussed in sections .3 and .6 .

the motion of the system's point. For example, consider a damped harmonic oscillator with F —k:r — ap, for which tlle velocity function is

$$\left(\frac{dx}{dt}, \frac{dp}{dt}\right) = \left(\frac{p}{m}, -kx - \alpha p\right).$$

A plot of this field for the undamped (a = O) and damped oscillators is

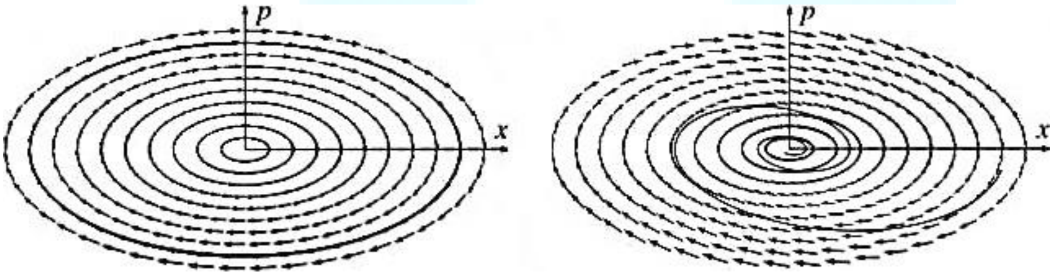

Figure 1.1: Velocity field for undamped and damped harmonic oscillators, and one possible phase curve for each system through phase space.

shown in Figure [J Tlle velocity field is everywhere tangent to any possible path, one of which is shown for each case. Note that qualitative features of the motion can be seen from the velocity field without any solving of the differential equations; it is clear tllat in the damped case the path of the system must spiral in toward t.lle origin.

The paths taken by possible physical motions through the phase space of an autonomous system have an important, property. Because the rate and direction with which the phase point moves away from a given point of phase space is completely determined by the velocity function at that point, if the system ever returns to a point it Illust move away from that point exactly as it did the last tirne. That is, if the syst.enl at time T returns to a point in phase space that it was at at time t O, then its subsequent motion must be just as it was, so -F t) = ü(t), and the motion is periodic with period T. This almost implies that the phase curve the Object takes through phase space must be nonintersectina

In the non-autonomous case, where the velocity field is time dependent, it may be preferable to think in terms Of extended phase space, a Gn + 1

[13] An exception can occur at an unstable equilibrium point, where the velocity function vanishes. The motion can just end at such a point, and several possible pluuse curves can terminate at that point.

dimensional space with coordinates (37, t). The velocity field can be extended to this space by giving each vector a last component of 1, as dt/dt 1. Then the motion of the systern is relentlessly upwards in this direction, though still complex in the Others. For the undamped one-dimensional harmonic oscillator, the path is a helix in the three dimensional extended phase space.

Most of this book is devoted to finding analytic methods for exploring the motion of a system. In several cases we will be able to find exact analytic solutions, but it should be noted that these exactly solvable problems, while very important, cover only a small set of real problems. It is therefore important to have methods other than searching for analytic solutions to deal with dynamical systems. Phase space provides one method for finding qualitative information about the solutions. Another approach is numerical. Nevvton's Law, and more generally the equation (ED for a dynamical system, is a set of ordinary differential equations for the evolution of the system's position ill phase space. Thus it is always subject 10 nulnerical soluLi011 given all initial configuration, at least up until such point that some singularity in the velocity function is reached. One primitive technique which will work for all such systems is to choose a small time interval of length At, and use dä/dt at the beginning of each interval to approximate Ari during this interval. This gives a new approximate value for at the end of this interval, which may then be taken as the beginning of the nextE

4This is a very unsophisticated method. The errors made in each step for AF and AP' are typically As any calculation of the evolution from time to to tf will involve a number (Itf — to]/At) of time steps which grows inversely to At, the cumulative error can be expected to be O (At). In principle therefore we can approach exact results for a finite time evolution by taking smaller and smaller time steps, but in practise there are other considerations, such as computer time and roundoff errors, which argue strongly in favor of using more sophisticated numerical techniques, with errors of higher order in At. Increasingly sophisticated methods can be generated which give cumulative errors of order $((\Delta t)^n)$, for any n. A very common technique is called fourth-order Rung&Kutta, which gives a,n error These methods can be found in any text on numerical methods.

As an example, we show the meat of a calculation for the damped harmonic oscillator. This same technique will work even with a very complicated situation. One need only add lines for all the components of the position and momenturn, and change the force law appropriately.

This is not to say that numeri- cal solution is a good way to solve


```
while (t < tf) { dx =
  (p/m) * dt; dp = —
  k*x+alpha*p)*dt ;
  x = x + dx;
  p = p + dp;
  t = t + dt;
  print t, x, p;
}
```

this problem. An analytical solu- Integrating the motion, for ation, if it can be found, is almost damped harmonic oscillator. always preferable, because

- It is far more likely to provide insight into the qualitative features of the motion.
- Numerical solutions must be done separately for each value of the parameters (k, m, Q) and each value of the initial conditions (xo and PO).
- Numerical solutions have subtle numerical problems in that they are only exact as At * O, and only if the computations are done exactly. Sometimes uncontrolled approximate solutions lead to surprisingly large errors.

Nonetheless, numerical solutions are often the only way to handle a real prob— lem, and there has been extensive development of techniques for efficiently and accurately handling the problem, which is essentially one of solving a system of first order ordinary differential equations.