Plasma Dispersion Relation and Instabilities in Electron Velocity Distribution Function

Introduction

In this overview paper we briefly describe methods of derivation and calculation of the dispersion relation for electromagnetic waves in a hot collisionless magnetized plasma.

We start with the dispersion relation of electrostatic waves in a hot unmagnetized plasma (or in a magnetized plasma with parallel propagation). Then we continue with a general case of the dispersion relation of electromagnetic waves in a hot plasma with an ambient magnetic field present. Also instabilities in electron velocity distribution function are discussed.

In the last part of the paper we discuss some applications, which include analysis of waveparticle interactions, mechanisms of wave generation and transfer of energy between different electron populations.

The intent of the paper is to be a summary of information the author came across while writing a code of a numeric dispersion relation solver, hence the new findings can be expected in subsequent papers.

Dispersion relation

Dispersion relation provides a relationship between the wave vector and the frequency of a wave and describes under which conditions the wave can propagate and under which conditions it cannot propagate. The dispersion relation can usually be obtained as a condition for non-trivial solutions of a homogeneous set of equations which describe given waves, and it is usually written in the form $D(\mathbf{k},\omega) = 0$. Equations describing a wave are usually in a form of a homogeneous set of partial differential equations, but after applying an integral transform (Fourier or Laplace) a set of algebraic equations has to be solved. Roots of the dispersion relation correspond to the actual modes of wave propagation.

The dispersion relation contains information not only on the wavelength and the frequency, but also on the direction of wave propagation. As the frequency is considered complex, the imaginary part of frequency $=\omega = \gamma$ describes the growth or damping rate of the wave, depending on the sign. For a given wave vector, generally more solutions of a complex frequency can exist. Phase and group velocities can be obtained from the dispersion relation.

The dispersion relation depends on the properties of a plasma, namely on phase space distribution functions of plasma particles, properties of plasma particles (mass and charge) and electric and magnetic field.

In order to be able to derive the dispersion relation for waves in a plasma, some assumptions are made. Hot homogeneous collisionless plasma in a magnetic field is assumed. The dispersion relation is derived using kinetic theory and linearization. It means that the starting point is a homogeneous (collisionless) version of the Boltzmann equation (also called the Vlasov equation)

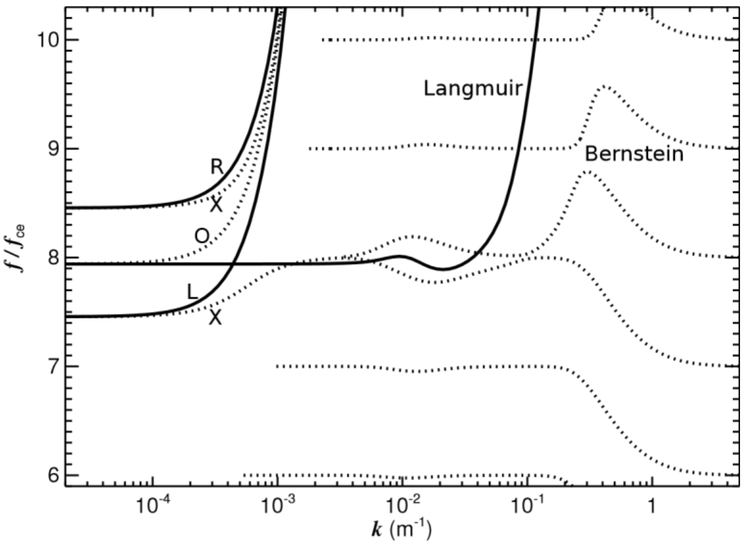


Figure 1. Example of the dispersion relation for two directions of wave propagation – parallel propagation (solid lines) and perpendicular propagation (dotted lines). Parallel to a magnetic field propagate electromagnetic R and L waves (the second R mode is not visible on the picture) and an electrostatic Langmuir mode wave. Perpendicular to a magnetic field propagate two extraordinary modes X, one ordinary mode O and electrostatic Bernstein modes for high values of k. Only real part of frequency is shown. The distribution function consists of three electron populations – a core population (n = 47.24cm^{-3}, v_{th} = 10eV, $T_\perp/T_k$ = 0.08), a warm population (n = 0.82cm^{-3}, v_{th} = 1180eV, $T_\perp/T_k$ = 0.98) and a warm electron population with a loss cone (n = 0.41cm^{-3}, v_{th} = 1890eV, $T_\perp/T_k$ = 1.16 and the loss-cone depth parameter β defined in *R¨onnmark* [1982] β = 0.2). Electron cyclotron and plasma frequencies are f_{ce} = 7875Hz and f_{pe} = 62505Hz. Parameters taken from *Grimald and Santol´ık* [2010].

with the set of Maxwell equations. Linearization means that varying quantities are expressed as a sum of a mean value and its first order perturbation. Solutions of waves are expected in a form of $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ (planar wave). As a consequence of a linearization, solutions are valid only during a linear phase until non-linear processes take over (a wave cannot grow indefinitely).

Example of the dispersion relation for a plasma consisting of several electron populations with the distribution function exhibiting the loss cone anisotropy for $\omega_{pe} > \omega_{ce}$ can be seen in Fig. 1.

Electrostatic waves

The starting point is the Vlasov equation

$$\frac{\mathrm{d}f_s(\mathbf{r},\mathbf{v},t)}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s = 0, \quad (1)$$

where f_s is the phase space distribution function of species s. By linearizing it ($f_s(\mathbf{v}) = f_0(\mathbf{v}) + f_1(\mathbf{v})$, f_1 is the first-order perturbation), $\mathbf{B} = 0$, with electric field being completely described by potential $\mathbf{E} = -\nabla\Phi$ and by performing a Fourier transform in both time and space, which brings us from spatial and temporal variables $\mathbf{r}$ and t to $\mathbf{k}$ and ω, yields the dispersion relation for electrostatic waves in a hot plasma without a magnetic field [*Gurnett and Bhattacharjee*, 2005]

$$D(k,\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial F_{s0}/\partial v_\parallel}{v_\parallel - \omega/k} \mathrm{d}v_\parallel = 0 \quad (2)$$

ω_{ps} denotes the plasma frequency of species s with an electric charge q_s, density n_s and a mass $m_s,\ \omega_{\rm ps}^2 = n_s q_s^2/(\varepsilon_0 m_s)$. We introduced the reduced distribution function $F_{s0}(v_k)$ = $1/n_0 \iint_{-\infty}^{\infty} f_0(\mathbf{v})\,\mathrm{d}v_x \mathrm{d}v_y$, such that $^{\mathrm{R}}F_{s0}\, dv_k = 1$.

This problem was solved by *Vlasov* [1938]. However, it has a problem when a resonance between the wave and particles moving at the phase speed of the wave occurs. It does not give us any advice on how to deal with the integral containing a pole. Therefore this equation can be solved only for the distribution function with no particles moving at the phase velocity of the wave. This is the reason this dispersion relation cannot lead to resonant instabilities.

Landau [1946] managed to solve the problem shortly after Vlasov. He uses Fourier transform in space, but Laplace transform in time. The dispersion relation looks the same as in the Vlasov case (2), but complex frequency ω is replaced by ip. It is not the final form, though. That form is valid only for the positive half of a complex plane p, $<p > 0$, and it needs to be analytically continued to the left half plane. The discontinuity when real part of p equals zero ($<p = \gamma = 0$) is related to the problem in the Vlasov approach.

The analytical continuation is performed by distorting the integration contour (originally running along the v_k axis from $-\infty$ to ∞) such that it always passes below the pole (see Fig. 2a). We integrate along the parallel velocity axis v_k, but we have to extend it to complex velocity so we can make a turn around the pole. Therefore in case of $\gamma < 0$ we have to calculate a residue. This involves evaluating the distribution function at a complex parallel speed. It is straightforward for analytically defined distribution function like Maxwellian, but not so obvious for the distribution function defined by data measured by a particle analyser on-board a spacecraft.

Having done the analytical continuation, the dispersion relation for electrostatic waves in a hot plasma without a magnetic field in case of $<p = \gamma > 0$ has the form from the Vlasov approach (2), but in case of $<p = \gamma < 0$ it is

$$D(k,p) = 1 - \sum_s \frac{\omega_{\rm ps}^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial F_{s0}/\partial v_\parallel}{v_\parallel - \mathrm{i}p/k}\,\mathrm{d}v_\parallel - 2\pi\mathrm{i}\frac{k}{|k|}\frac{\omega_{\rm ps}^2}{k^2}\left.\frac{\partial F_{s0}}{\partial v_\parallel}\right|_{v_\parallel = v_{\rm Res} = \frac{\mathrm{i}p}{k}} = 0, \tag{3}$$

and it can be written more generally as

$$D(k,p) = 1 - \sum_s \frac{\omega_{\rm ps}^2}{k^2} \int_C \frac{\partial F_{s0}/\partial v_\parallel}{v_\parallel - \mathrm{i}p/k}\,\mathrm{d}v_\parallel = 0, \tag{4}$$

given that the contour C runs along the v_k axis and always stays below the pole.

The complex residue contributes to the imaginary part of the dispersion relation $=D$ and was lacking in the Vlasov approach (the real part of the dispersion relation $<D$ in the first order approximation equals the dispersion relation obtained in the Vlasov approach (2)). Shortly we will see that the imaginary part of the dispersion relation $=D$ is responsible for resonant growth (damping).

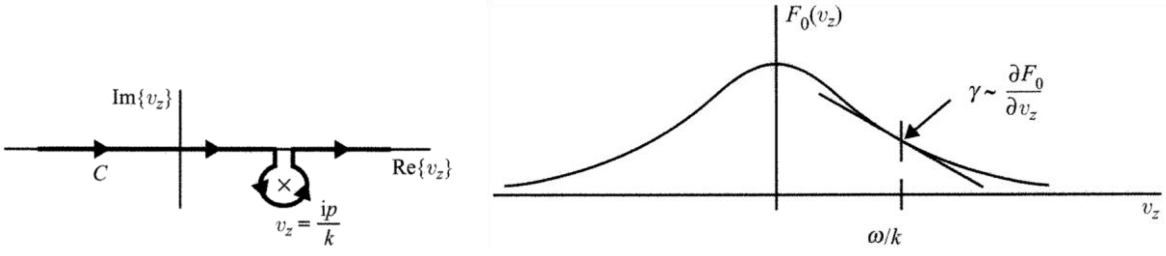
When assuming the small growth rate $|\gamma| \ll |\omega|$ it is possible to derive a formula for the growth rate for a given wave. Using the Taylor expansion of $D(k,p)$ around $p_0 = <p = -\mathrm{i}\omega$ and stating $D = 0$ leads to the expression for the growth rate

$$\gamma = \frac{-\Im D}{\partial \Re D/\partial \omega}, \tag{5}$$

which, after expressing $=D$, leads to

$$\gamma = \pi \frac{k}{|k|}\frac{\omega_{\rm ps}^2}{k^2 \partial \Re D/\partial\omega}\left.\frac{\partial F_{s0}}{\partial v_\parallel}\right|_{v_\parallel = \omega/k}. \tag{6}$$

The growth rate is proportional to the slope of the reduced distribution function evaluated at the phase velocity of the wave. Specifically for Langmuir waves it can be shown that the $\partial < D/\partial\omega$ term is always positive and for velocity distributions with a single maximum centered at zero the growth rate is always negative (because of the negative slope, see Fig. 2b). This is the Landau damping. The energy of the wave is transfered into the energy of particles. This



(a) (b)

Figure 2. (a) Distorted integration contour in a complex velocity plane so it always passes below the pole at $v_k = \mathrm{i}p/k$ (diagram for $k > 0$; for $k < 0$ it would be made to pass above the pole). (b) Assuming the weak growth rate approximation ($|\gamma| \ll |\omega|$), the growth rate γ is proportional to the slope of the reduced distribution function evaluated at the phase velocity $v_k = \omega/k$ of the wave. Images from *Gurnett and Bhattacharjee* [2005], v_z corresponds to v_k.

discovery was originally a bit surprising for a collisionless plasma. Contrary to equations for a plasma with collisions, the Vlasov and Poisson's equations are reversible in time, but lead to irreversible processes [*Mouhot and Villani*, 2010].

For a plasma with Maxwellian distribution functions

$$F_{s0}(v_\parallel) = \frac{1}{\sqrt{\pi} v_{\mathrm{th}s}} \exp\left[-\left(\frac{v_\parallel}{v_{\mathrm{th}s}} - v_{\mathrm{d}s}\right)^2\right] \tag{7}$$

with $v_{\mathrm{th}s} = \sqrt{\frac{2k_\mathrm{B}T_s}{m_s}}$ being thermal velocity of species s and $v_{\mathrm{d}s}$ being drift velocity normalized by $v_{\mathrm{th}s}$, the dispersion relation (4) is often expressed using the so called plasma dispersion function

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_C \frac{e^{-z^2}}{z - \zeta}\, \mathrm{d}z, \tag{8}$$

where the integral is calculated along the contour C defined the same way as above. The dispersion relation (4) can then be written as

$$D(k,p) = 1 + \sum_s \frac{1}{(k\lambda_{\mathrm{D}}s)^2}\left[1 + \zeta_s Z(\zeta_s)\right] = 0 \quad , \tag{9}$$

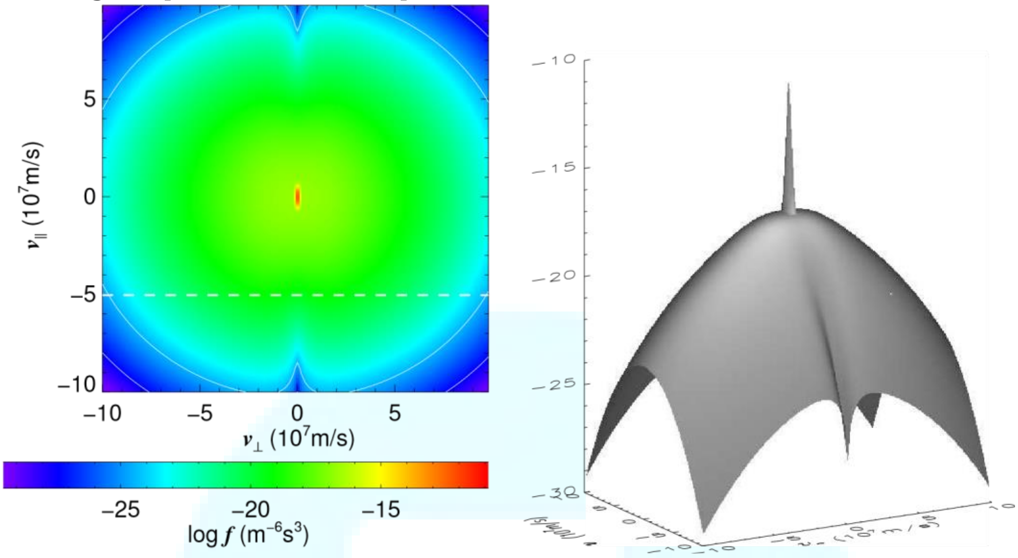
where $\zeta_s = \frac{1}{v_{\mathrm{th}s}}\,\mathrm{i}p/k$ and $\lambda_{\mathrm{D}s}$ is the Debye length $\lambda_{\mathrm{D}s} = \sqrt{\frac{\epsilon_0 k_\mathrm{B} T_s}{n_s q_s^2}}$. The advantage of introducing the plasma dispersion function $Z(\zeta)$ is also that it is already implemented in many numerical packages (often by the so-called Faddeeva function $w(\zeta)$, $Z(\zeta) = \mathrm{i}\sqrt{\pi}w(\zeta)$) and one does not need to deal with complex integration along a contour.

Electromagnetic waves

The wave equation of electromagnetic waves

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E})+\frac{\omega^2}{c^2}\mathbf{K}\cdot\mathbf{E}=0 , \quad (10)$$

where **K** is the dielectric tensor $\mathbf{K}=\mathbf{1}-\frac{\sigma}{\mathrm{i}\omega\epsilon_0}$ and σ the conductivity tensor, can be expressed as a dot product of the dispersion tensor and the electric field, $\mathbf{D}\cdot\mathbf{E}=0$. The dispersion relation is defined as a requirement for non-trivial solutions of **E**, which means that the determinant of the dielectric tensor **D** equals zero, det**D** = 0. Calculating the actual modes of the dispersion relation involves finding complex roots of a complex function det**D**.



(a) (b)

Figure 3. Example of the distribution function consisting of three electron populations (a core population of cold electrons and two populations of hot electrons with bi-Maxwellian distributions) exhibiting the loss-cone anisotropy (the third electron population), color mapped graph (a) and as a 3D graph (b). $v_\perp$ and $v_\parallel$ axes in panel (b) span across the same range as in (a). The electron distribution function corresponds to electrons trapped in the Earth's magnetosphere.

In order to compute the dielectric tensor the linearized Vlasov (zero- and first-order) and Poisson's equations are taken with new terms containing magnetic field [*Gurnett and Bhattacharjee*, 2005]. The dielectric tensor can be expressed as [*R¨onnmark*, 1982; *Treumann and Baumjohann*, 1997b]

$$\mathbf{K}(\mathbf{k},\omega)=\left(1-\sum_s\frac{\omega_{ps}^2}{\omega^2}\right)\mathbf{I}-\sum_s\sum_{n=-\infty}^{\infty}\frac{\omega_{ps}^2}{\omega^2}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{k_\parallel\frac{\partial F_{s0}}{\partial v_\parallel}+\frac{n\Omega_s}{v_\perp}\frac{\partial F_{s0}}{\partial v_\perp}}{k_\parallel v_\parallel+n\Omega_s-\omega}S_{ns}\;2\pi v_\perp\,\mathrm{d}v_\perp\mathrm{d}v_\parallel, \quad (11)$$

$$S_{ns}(v_\perp,v_\parallel)=\begin{pmatrix}\frac{n^2v_\perp^2}{\beta_s^2}J_n^2 & \mathrm{i}\frac{nv_\perp^2}{\beta_s}J_nJ_n' & \frac{nv_\perp v_\parallel}{\beta_s}J_n^2\\ -\mathrm{i}\frac{nv_\perp^2}{\beta_s}J_nJ_n' & v_\perp^2J_n'^2 & -\mathrm{i}v_\perp v_\parallel J_nJ_n'\\ \frac{nv_\perp v_\parallel}{\beta_s}J_n^2 & \mathrm{i}v_\perp v_\parallel J_nJ_n' & v_\parallel^2J_n^2\end{pmatrix}, \quad (12)$$

where $\beta_s = k_\perp v_\perp/\Omega_s$ and Ω_s is the gyrofrequency of species s defined by $\Omega_s = q_sB/m_s$. J_n and J_n' denotes Bessel function of the first kind and its derivative respectively and is a function of β_s. From the computational point of view calculating the dielectric tensor is the most difficult part. It simplifies for some special conditions, specifically for parallel propagation only one component of S_{ns} tensor has to be computed for the electrostatic mode and two components for the electromagnetic R and L modes with the other components being zero. For general oblique wave propagation all components of S_{ns} tensor have to be computed.

Wave instabilities

Considering the physical mechanism involved, instabilities can be divided into reactive (non-resonant or fluid type) and resonant or kinetic instabilities.

Reactive instability is a bunching instability [*Cairns and Fung*, 1988], where growing waves create bunches of electrons in space, which enhance electric field. It leads to quasimonochromatic, very narrow band waves.

Kinetic instability is a resonant instability and involves interaction of the wave with particles, which move at speeds close to the wave phase velocity. It may lead to narrow or broad band waves. Examples include inverse Landau or cyclotron damping.

Unstable (resonant) electron distribution function must have more than one hump [*Gurnett and Bhattacharjee*, 2005]. This usually means at least two electron (or ion) components (e.g. core and a beam population) [*Treumann and Baumjohann*, 1997a] or for example the distribution function anisotropy (see Fig. 3 for the loss-cone anisotropy). It is valid for both electrostatic and electromagnetic waves with the difference that in case of electrostatic waves only the electric field is involved in resonant interaction, but in the case of electromagnetic waves both the electric and magnetic fields are involved. Sometimes the magnetic field plays the dominant role [*Gurnett and Bhattacharjee*, 2005].