Polarization

1 Polarization vectors

In the last lecture, we showed that Maxwell's equations admit plane wave solutions

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}, \quad \vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} \tag{1}$$

Here, $\vec{E}_0$ and $\tilde{B}_0$ are called the **polarization vectors** for the electric and magnetic fields. These are complex 3 dimensional $\omega = c|\vec{k}|$ vectors. The wavevector $\tilde{k}$ and angular frequency ω are real and in the vacuum are related by. This relation implies that electromagnetic waves are dispersionless with velocity c: the speed of light. In materials, like a prism, light can have dispersion.We will come to this later.

In addition, we found that for plane waves

$$\vec{B}_0 = \frac{1}{\omega}(\vec{k} \times \vec{E}_0) \tag{2}$$

This equation implies that the magnetic field in a plane wave is completely determined by the electric field. In particular, it implies that their magnitudes are related by

$$|\vec{E}_0| = c|\vec{B}_0| \tag{3}$$

and that

$$\vec{k}\cdot\vec{E}_0 = 0, \quad \vec{k}\cdot\vec{B}_0 = 0, \quad \vec{E}_0\cdot\vec{B}_0 = 0 \tag{4}$$

In other words, the polarization vector of the electric field, the polarization vector of the magnetic field, and the direction $\tilde{k}$ that the plane wave is propagating are all orthogonal.

To see how much freedom there is left in the plane wave, it's helpful to choose coordinates. We can always define the $\hat{z}$ direction as where $\tilde{k}$ points. When we put a hat on a vector, it means the unit vector pointing in that direction, that is $\hat{z}=(0, 0, 1)$. Thus the electric field has the form

$$\vec{E} = \vec{E}_0 e^{i\omega(\frac{z}{c} - t)} \tag{5}$$

which moves in the z direction at the speed of light. Since $\tilde{E}_0$ is orthogonal to $\tilde{k}$ we can write also write

$$\tilde{E}_0 = (E_x, E_y, 0) \tag{6}$$

with E_x and E_y complex numbers.

The two complex amplitudes E_x and E_y each have magnitudes and phases. We can write these explicitly as $E_x = |E_x|e^{i\varphi_x}$ and $E_y = |E_y|e^{i\varphi_y}$. So one needs four real numbers to specify the polarization vector. We usually separte the phases into an overall phase and the difference in phase between E_x and E_y, $\varphi = \varphi_x - \varphi_y$. The reason for doing this is that as t and z change the phases of E_x and E_y change in the same way while φ is unaffected. We don't usually care about this overall phase, or about the overall magnitude $E = |E_x|^2 + |E_y|^2$. Thus, to specify a polarization vector, we talk about the relative size of E_x and E_y and the phase difference $\varphi = \varphi_x - \varphi_y$. Don't let the complex polarization vectors confuse you. We can just as well

$$\tilde{E} = E_x\hat{x}\cos(kz - \omega t + \varphi_x) + E_y\hat{y}\cos(kz - \omega t + \varphi_y) \tag{7}$$

where you see the four real numbers specifying the polarization explicitly. Exponentials just make a lot of the algebra easier.

2 Linear polarization

We say a plane wave is **linearly polarized** if there is no phase difference between E_x and E_y. We can write linear polarizations as

$$\tilde{E}_0=(E_x,E_y,0) \quad (8)$$

and choose the overall phase so that E_x and E_y are real numbers. If E_y=0 but $E_x \neq 0$, we have

$$\tilde{E}=E_0\hat{x}e^{i(kz-\omega t)} \quad (9)$$

with $E_0=|\vec{E}_0|$ just a number now. Then, from Eq. (2), since $\hat{z}\times\hat{x}=\hat{y}$,

$$\vec{B}=\frac{1}{c}E_0\,\hat{y}e^{i(kz-\omega t)} \quad (10)$$

This configuration is said to be **linear polarized in the x direction**. Similarly we can have

$$\tilde{E}=E_0\hat{y}e^{i(kz-\omega t)} \text{ and} \quad (11)$$

using $\hat{z}\times\hat{y}=-\hat{x}$:

$$\vec{B}=-\frac{1}{c}E_0\hat{x}e^{i(kz-\omega t)} \quad (12)$$

This configuration is said to be **linear polarized in the y direction**. Note that in both cases, the magnetic field is given by rotating the electric field 90° counterclockwise in the *x*-*y* plane and dividing by *c*.

More generally, for any unit vector $\hat{v}$ we can have

$$\vec{E}=E_0\hat{v}\,e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (13)$$

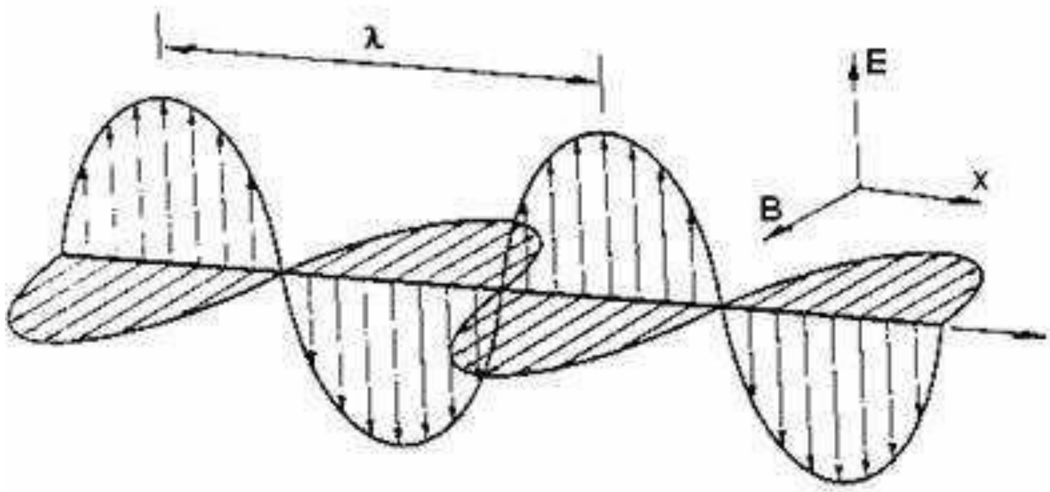
which is linearly polarized in the $\hat{v}$ direction. The magnetic field will be polarized in direction 90° behind the electric field.

Remember, we always implicitly want to have the real part of these fields. So, for linearly polarized light in the *x* direction, the fields are actually

$$\mathrm{Re}[\vec{E}]=(E_0\cos(kz-\omega t),0,0), \qquad \mathrm{Re}[\vec{B}]=\left(0,\frac{E_0}{c}\cos(kz-\omega t),0\right) \quad (14)$$

Note that there is no *x* or *y* dependence in these solutions: the fields are completely uniform in *x* and *y* – they are plane waves. At each point on the plane, the electric field points in the $\hat{x}$ direction with the same magnitude. This magnitude varies as we move in *z* and in *t* but is always uniform in the plane.

Here's a picture of how the fields vary as they move along for a wave moving in the *x* direction:



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Circular polarization

3 Circular polarization

What if the components of the electric field are not in phase? First suppose the have the same magnitude but are a quarter wavelength out of phase, so $\phi_x - \phi_y = \frac{\pi}{2}$. Then,

$$\vec{E}_0 = \left(E_0, E_0 e^{i\frac{\pi}{2}}, 0\right) = (E_0, i E_0, 0)$$

$$\vec{E} = \left(E_0\, e^{i(kz-\omega t)}, i E_0\, e^{i(kz-\omega t)}, 0\right)$$

(15) Thus,

(16)

Taking the real part gives the actual electric field

$$\text{Re}[\vec{E}] = \begin{pmatrix} E_0 \cos(kz - \omega t), & -E_0 \sin(kz - \omega t), & 0 \end{pmatrix} \quad (17)$$

This is called **left-handed circularly polarized light.**

What does it look like? Well, at $t = z = 0$, the field is Re $(\vec{E}) = (E_0, 0, 0)$ pointing in the x direction. A little farther along, when $kz = \frac{\pi}{2}$, still at $t = 0$, then Re $(E) = (0, -E_0, 0)$ which points in the negative $\hat{y}$ direction. Farther along still, when $kz = \pi$, it points along $-\hat{x}$, and then $\hat{y}$ and so on. Finally, after a full wavelength, it goes back to $\hat{x}$. Thus, as we move along z, the polarization rotates clockwise in the x-y plane. Equivalently, at a given z, as time progresses it also rotates in the x-y plane.

Here is a picture

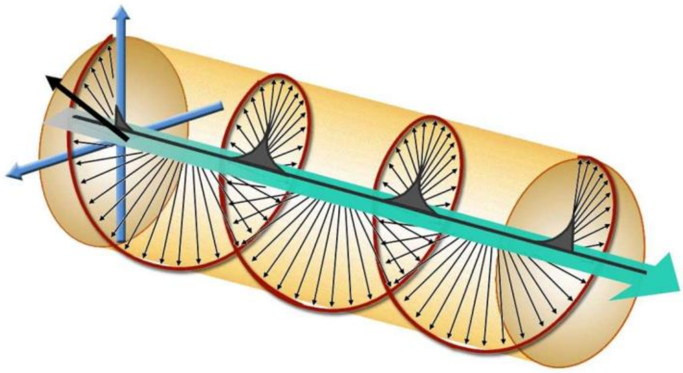

Figure 1. Circularly polarized light changes the direction of its polarization as it moves.

Similarly, taking $\phi_x - \phi_y = -\frac{\pi}{2}$ gives $\vec{E}_0 = (E_0, -i E_0, 0)$

$$\text{Re}[\vec{E}] = (E_0 \cos(kz - \omega t), E_0 \sin(kz - \omega t), 0) \quad (18)$$

which is **right-handed circularly polarized light**. In this case, at t=0 and kz=0, the polarization points in the $\hat{x}$ direction. A quarter wavelength farther, it points in the $\hat{y}$ direction, and so one. So this field rotates counterclockwise in the x-y plane.

Note that when we add left and right handed polarizations we get

$$\vec{E}_0 = (E_0, i E_0, 0) + (E_0, -i E_0, 0) = (2E_0, 0, 0) \quad (19)$$

which is linearly polarized in the $\hat{x}$ direction. Similarly, subtracting them gives linear polarization in the $\hat{y}$ direction. Thus circular and linear polarizations are not linearly independent. Indeed, any possible polarization can be written as a linearly combination of left-handed and right-handed circularly polarized light.

Circularly polarized light has angular momentum, $\pm e_0 |\vec{E}|^2 \hat{k}$ with the positive sign for lefthanded and the negative sign for right-handed. Linearly polarized light carries no angularmomentum. This is easy to understand because it is a sum of left and right handed light. As an analogy, suppose you have two tops, one spinning clockwise and one spinning counterclockwise; the two-top system has no net angular momentum.

As mentioned before, light cannot have arbitrarily small intensity. The smallest intensity light can have is a single photon. Thus the photon itself must be polarized. A single circularly polarized photon has angular momentum $\vec{J} = \pm\frac{h}{2\pi}\hat{k}$. It may seem surprising that photons which are pointlike particles with no substructure can have angular momentum on their own. It is a very odd fact, but true nonetheless. Photons have **spin**. We say photons are particles of spin 1. The two signs for the angular momentum correspond to the two helicities of the photon.

To complete the discussion of polarization, we can consider also varying the magnitudes of the different components and the phase. The most general parameterization is

$$\vec{E}_0 = E_x\hat{x}\cos(kz-\omega t) + E_y\hat{y}\cos(kz-\omega t+\phi) \tag{20}$$

When E_x and E_y are different and the relative phase φ is nonzero, the polarization changes in magnitude as it rotates in the x-y plane. It thereby describes an ellipse, and this is called **elliptical polarization**. Linear polarization corresponds to $\varphi = 0$. Circular polarization corresponds to $\phi = \pm\frac{\pi}{2}$ and $E_x = E_y$.

4 Polaroid film

Now that we understand the mathematics of polarizations, what is the physics? How do we actually produce polarized light?

One way to polarize light is using a **polaroid film**. The first such film was invented by Edwin Land in 1928, while an undergraduate at Harvard. He came up with a way to align polymer molecules in a thin sheet into long needlelike strands. When an electric field acts on the electrons in those strands, it can only move the electrons up and down in the strand direction, but not perpendicular to the strands. Thus the only polarization which can pass through is linearly polarized **in the direction perpendicular to the strips**. Here is a figure

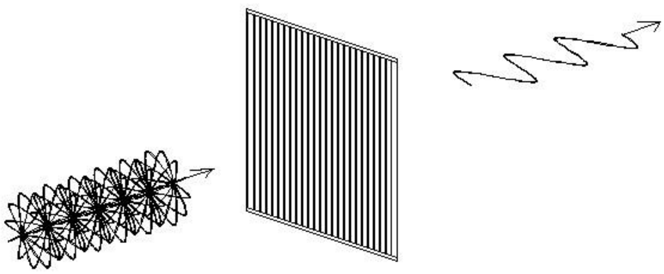

Figure 2. Polarizing film absorbs the electric field in the direction of the strips.

Land called his invention **Polaroid film**, patented it, and then founded the Polaroid corporation here in Cambridge. One of the primary applications at the time was sunglasses (see below). His next most famous invention was the Land instant camera, now known as the polaroid camera. The polaroid camera has nothing to do with polarization!

Back to polarizing film. Say the film has the strips in the y direction and a plane wave comes in along the z direction with an arbitrary polarization

$$\vec{E}_{\text{init}} = E_x\hat{x}e^{i(kz-\omega t)} + E_y\hat{y}e^{i(kz-\omega t+\varphi)} \tag{21}$$

Since the polarizer has strips in the y direction, it absorbs the $\hat{y}$ components of the electric field. Thus, what exits the polarizer must be

$$\vec{E}_{\text{final}} = E_x\hat{x}\, e^{i(kz-\omega t)} \tag{22}$$

Thus the polarizer takes any initial polarization and turns in into linear polarization in the x direction.

Polarization in the real world

For example, say the field is linearly polarized at an angle θ to the direction where the field would just go through. That is,

$$\vec{E}_{\text{init}} = E_0(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})e^{i(kz-\omega t)} \quad (23)$$

Then if we put it through the polarizer with strips in the y direction we would find

$$\vec{E}_{\text{init}} = E_0 \cos\theta\, \hat{x} e^{i(kz-\omega t)} \quad (24)$$

So that the result is still linearly polarized but now in the $\hat{x}$ direction with magnitude $E_0\cos\theta$. Since the intensity of radiation is proportional to the square of the field, we would find that a polarizer oriented at an angle θ to the direction of propagation reduces the intensity

$$I_{\text{final}} = I_{\text{initial}} \cos^2\theta \quad (25)$$

This is known as **Malus' law**.

5 Polarization in the real world

The key facts which let us understand why natural light is polarized are 1) that the electric field makes charged particles move in the direction of $\vec{E}$ and 2) moving charged particles produce electric fields along their direction of motion. Fact 1) and the consequent absorption of energy of the electric field is the mechanism behind the polaroid film we just discussed.

During the day, sunlight is constantly being absorbed and emitted by molecules (mostly water) in the atmosphere. Let's say z is the vertical direction (towards the sky), so the sky above us is the x-y plane. Let's say the sun is in the x direction. Thus plane waves from the sun have $\vec{k}$ in the $\hat{x}$ direction and are polarized in the y-z plane. Thus, the sky molecules can only get pushed in the y and z directions from sunlight. Molecules moving in the z direction emit light in the x-y plane, that is, off into the sky, but not to us. Molecules excited in the y direction however can emit light going directly off in z down to us. Since all this light comes from motion in the y-z plane, the light will all be linearly polarized in this plane. Thus the sky directly above us, at right angles to the sun, is polarized.

Similar arguments show that light from some other angle will have some polarization, but not be completely polarized. Looking through the sky directly at the sun, the light should not be polarized at all.

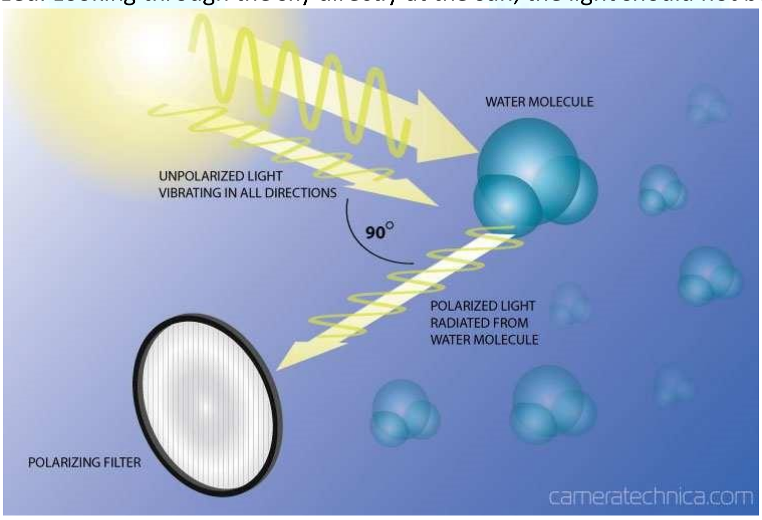

Figure 3. Polarization of the sky

Another important example is light from a reflection. Here the story is nearly identical, as shown in Fig 4 with the maximum polarization coming if the source is at a right angle to the viewer. In this case, the index of refraction of the two materials involved (air and lake for example) play a role and the angle is not exactly 90 degrees. Instead it is called Brewster's angle. We will derive a formula for Brewster's angle in Lecture 16 on reflection.

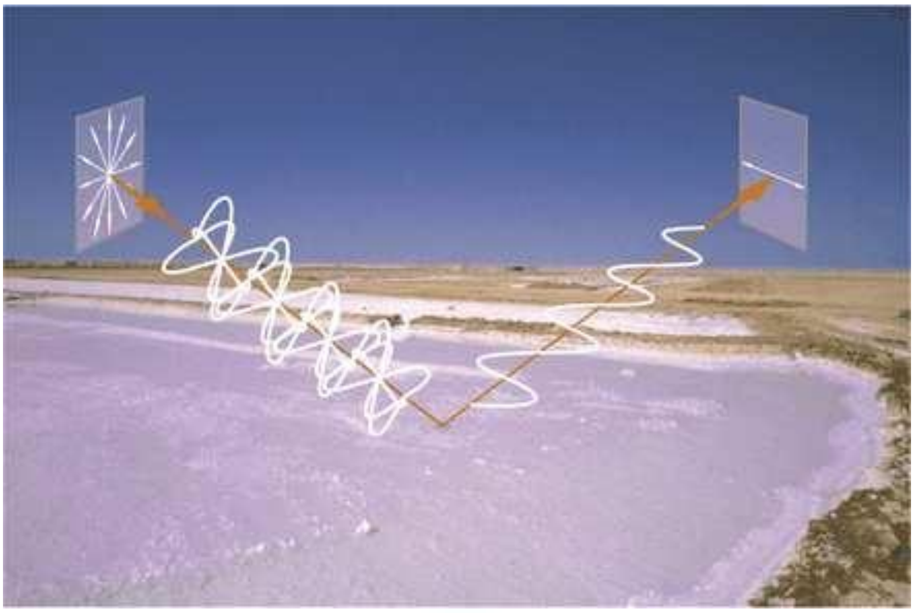

Figure 4. Polarization from reflection.

So reflected light is polarized. Most light around us is not polarized. So for example, if you are on a boat and light is coming in from the sky (not just the sky directly above you), and also from the water, only the light reflected off the water will be polarized. Thus if you put on polarizing sunglasses, you can filter out the reflected light, and see more clearly the things around you. Similarly, reflected light coming off the ground will be polarized horizontally. Polarizing sunglasses generally try to filter out horizontally polarized light.

6 Index of refraction

The speed of light c is the speed of light in the vacuum. In air, light goes slightly slower than c. More generally, light travels at a speed

$$v = \frac{1}{n}c \tag{26}$$

with n called the **index of refraction**. The physical origin of the index of refraction is interesting. Light comes in and excites molecules. These molecules then radiate more light which interferes with the incoming light. The phases of the incoming and radiated light align in such a way as to make the light appear to be slower. We will work this out in detail in Lecture 17. For now, let's accept the index of refraction as as a phenomenological fact.

When light hits a boundary where the index of refraction changes, we have to solve the wave equation with the boundary conditions. We will work out the transmission and reflection coefficients (which are polarization dependent) in Lecture 16. All we need to know for know is that the incident and transmitted wave have the same frequency (this is always true – the boundary knows about the frequency but not the wavelength of the incoming wave).

Say light with frequency ω goes from a vacuum with n = 1 to a region with n = 2 and then back to the vacuum with n=1. Since the frequency is fixed and

$$\omega = vk = \frac{c}{n}\frac{2\pi}{\lambda} \tag{27}$$

we must have

$$n_1\lambda_1 = n_2\lambda_2 \tag{28}$$

So that as n goes up (at fixed frequency), λ must go down. The picture at a fixed time looks like this:

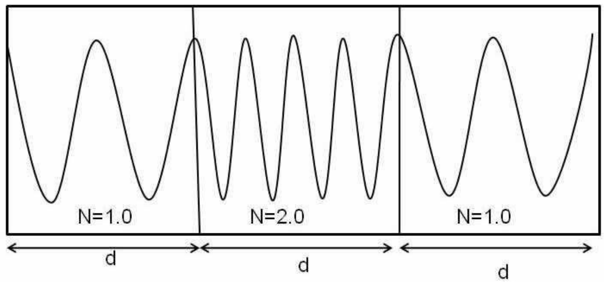

Figure 5. Plane wave entering and emerging from a medium with different index of refraction. This figure illustrates how light going slower means it goes through more oscillations over the same distance.

Many materials are **birefringent**, meaning they have different indices of refraction in different directions. A linear polarizer is an extreme example: it has $n=\infty$ in one direction, so that light simply does not propagate for one polarization. The mineral calcite is a naturally occurring birefringent material. Calcite's two indices of refraction are

$$n_1=1.66, \qquad n_2=1.49 \tag{29}$$

These indices of refraction are determined by the atomic structure of the mineral and associated with the primary axes of the crystal structure. You can see the birefringence by looking at some writing through a chunk of calcite. You will see two images. The two images are due to the different polarizations of light refracting differently in the crystal (refraction is the topic of Lecture 16). As you rotate the crystal, you can see how one polarization is refracting more than the other.

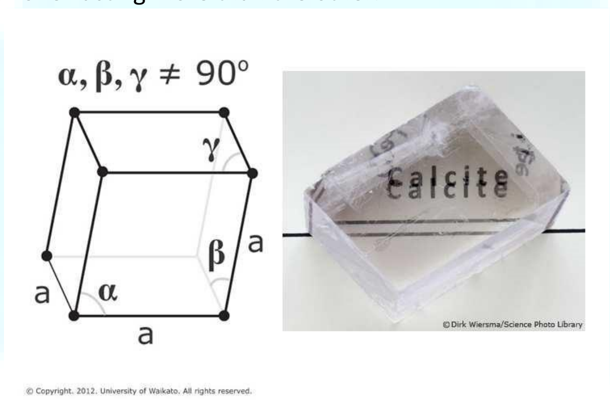

Figure 6. Calcite is naturally birefringent. When looking through a calcite crystal you see two images because the two polarizations of light refract differently.

7 Quarter and half-wave plates

Suppose we have a plane wave moving in the $\hat{z}$ direction whose electric field is polarized in the $\hat{x}+\hat{y}$ direction. Say this wave passes through a calcite crystal of thickness L whose optical axes are aligned with the $\hat{x}$ and $\hat{y}$ directions. The initial electric field is

$$\vec{E}_{\text{init}}(z,t) = E_0\,(\hat{x}+\hat{y})\,e^{i\omega\left(t-\frac{n}{c}z\right)} \tag{30}$$

where n is the index of refraction in air. Inside the crystal, the two components propagate with different speeds, so (assuming zero reflection)

$$\tilde{E}_{\text{inside}}(z,t) = E_0\,\hat{x}\,e^{i\omega\left(t-\frac{n_x}{c}z\right)} + E_0\hat{y}\,e^{i\omega\left(t-\frac{n_y}{c}z\right)} \tag{31}$$

where n_x and n_y are the two indices of refraction of the calcite.

In particular, at $z = L$ we have

$$\vec{E}_{\text{inside}}(L,t) = E_0\,\hat{x}\,e^{i\omega(t-\frac{n_x}{c}L)} + E_0\hat{y}\,e^{i\omega(t-\frac{n_y}{c}L)} \tag{32}$$

Thus there is a net phase difference of

$$\Delta\phi = \frac{\omega}{c}L(n_x - n_y) \tag{33}$$

between the x and y components of a field. Then when the light exits and both indices of refraction are n again, the outgoing wave has electric field

$$\vec{E}_{\text{final}} = E_0\,(\hat{x} + \hat{y}e^{i\Delta\phi})\,e^{i\omega(t-\frac{n}{c}z)} \tag{34}$$

From this calculation, we can now deduce how to use calcite to turn linearly polarized light into circularly polarized light. To do so, we must choose the length L so that $\Delta\phi = \frac{\pi}{2}$. This is easy to do if we know n_x and n_y using Eq. (33). A material which rotates the phase by $\frac{\pi}{2}$, which is a quarter of a circle, is known as a **quarter-wave plate**.

A few things to note about quarter wave plates or more generally using birefringent materials to polarize light:

- The phase $\Delta\varphi$ depends linearly on the frequency ω. So you need different lengths for different frequencies. In most materials, n_x and n_y also depend on ω.
- Light which is linearly polarized along one of the axes of the crystal will remain linearly polarized. That is, the orientation of the quarter wave plate is important.

In practice, if one has a coherent, monochromatic, polarized source, such as a laser, one knows the frequency and the direction of polarization. Then one can rotate between linear and circular polarization using wave plates.

If one places two quarter-wave plates together, or equivalently takes a crystal or length $2L$, one gets a **half-wave plate**. The first L of it adds a phase $e^{i\frac{\pi}{2}} = i$, rotating linear polarization to circular polarization; the second L adds another phase $e^{i\frac{\pi}{2}} = i$, rotating circular back to linear. However, the net phase changes is $e^{i\pi} = -1$ so $(E_x, E_y) \to (E_x, -E_y)$ and the final polarization is now rotated 90° from the original polarization. Thus half-wave plates can rotate the polarization of linearly polarized light. A crystal of length $4L$ will have $\Delta\varphi = 2\pi$ and return the polarization to its initial value. This is called a **full-wave plate**.

One application of the use of circularly polarized light is in 3D movies. Ever wonder what those glasses they give you at IMAX do? The IMAX theater projects two movies at once, one with left-hand circular polarization and the other with right-handed circular polarization. Your 3D glasses have one lens which can see transmit the left-handed light and the other lens transmits only right-handed. Thus you see different images with each eye which gives the illusion of depth. Why do you think they don't use linearly polarized light?

For another application, consider that the index of refraction along axes in a some materials can be controlled by applying electric and magnetic fields. In the case of electric fields, this effect is called **electro-optic effect**. Then the index of refraction is $n(\vec{E})$. In the case of magnetic fields, this effect is called the Faraday effect, and the index of refraction is $n(\vec{B})$. These effects are exploited in electro-optic devices which allow fast and precise control of the polarization of light.

An example of an electro-optic device is the rapatronic camera. In a ordinary camera, where the shutter is mechanical, the shutter speed is limited to about 1 millisecond. In a rapatronic camer, the shutter is composed of two crossed polarizers with a birefringent material exhibiting the electro-optic effect in between the polarizers. With the voltage on, the material acts as a half-wave plate so that all the light is transmitted through the shutter. If the voltage is switched off, which can be done on the order of microseconds or faster, , the material has no effect on the polarization. This causes the shutter to be closed. Rapatronic cameras were heavily developed during the 1940's to take pictures of atomic bomb detonations very shortly ($10^{-6}\,s$) after the detonation. This allowed scientists to learn about how well the bomb, in particular its detonation, was working. Here's a picture from a rapatronic camera:

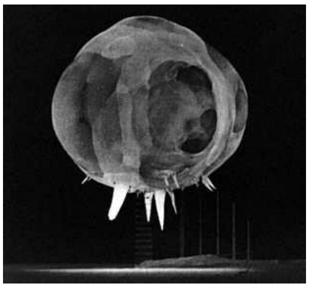

Figure 7. Picture of the detonation of an atomic bomb exposed with a rapatronic camera for 3 millionths of a second.