Radiation by Moving Charges

8.1 Introduction

The problem of radiation of electromagnetic waves by a single charged particle moving at an arbitrary velocity had correctly been formulated independently by Lienard and Wiechert before the advent of the special relativity theory. This is because once emitted from a charged particle, electromagnetic waves propagate at the speed *c* irrespective of the velocity of the charged particle, just as sound waves propagate at a speed independent of the source velocity. (The major nding made by Einstein was that electromagnetic waves still propagate at the speed *c regardless of the observer s velocity in contrast to the case of sound waves*.)

The scalar and vector potentials due to a moving charge can be found rigorously using the Green s function for the wave equation. Then radiation electromagnetic elds can readily be calculated. In nonrelativistic regime, the radiation power only depends on the acceleration of charged particles. As the velocity approaches *c;* however, signi cant increase in the radiation power occurs. Furthermore, in highly relativistic limit, radiation occurs primarily along the direction of velocity within an angular spread of order ' 1= about the velocity irrespective of the direction of ac-

celeration. Here = 1=p1 $_2$ is the relativity factor with = *v=c:* Hence radiation frequency is subject to strong Doppler shift. For example, in synchrotron radiation due to highly relativistic electron beam bent or undulated by a magnetic eld, radiation even in hard x-ray regime can be created.

In material medium, radiation processes without acceleration on charged particles are possible. If the velocity of a charged particle exceeds the velocity of electromagnetic waves in the medium

$$v > \frac{1}{\sqrt{\varepsilon \mu_0}},$$

where ″ is the permittivity, Cherenkov radiation occurs. Furthermore, if a charged particle crosses a boundary of two dielectric media, the transition radiation occurs even if the condition for Cherenkov radiation is not met. Transition radiation is due to sudden change in the normalized velocity from $_1$ = vp″$_1$ $_0$ to $_2$ = vp″$_{2\,0}$ which may be regarded as an e⁄ective acceleration even though the particle velicty *v* remains constant.

ENTRI

8.2 Lienard-Wiechert Potentials

The charge and current densities of a moving point charge are singular and described by

$$(\mathbf{r};t) = e\,[\mathbf{r} \qquad \mathbf{r}_p(t)]; \tag{8.1}$$

$$\mathbf{J}(\mathbf{r};t) = e\mathbf{v}(t)\,[\mathbf{r} \qquad \mathbf{r}_p(t)]; \tag{8.2}$$

where e is the charge, $\mathbf{r}_p(t)$ is the instantaneous location of the charge and $\mathbf{v}(t) = d\mathbf{r}_p(t)=dt$ is the instantaneous velocity of the charge which may be changing with time. Exploiting the Green s function for the wave equation,

$$G(\mathbf{r} \qquad \mathbf{r}', t-t') = \frac{1}{4\pi\,|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right), \tag{8.3}$$

we can write down solutions for the inhomogeneous wave equations,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{\rho}{\varepsilon_0}, \tag{8.4}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad \mathbf{A} = \qquad {}_0\mathbf{J}; \tag{8.5}$$

in the form

$$\Phi(\mathbf{r},t) = \frac{e}{4\pi\varepsilon_0}\int dV'\int dt'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta[\mathbf{r}'-\mathbf{r}_p(t')]\delta[f(t')],$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 e}{4\pi}\int dV'\int dt'\frac{\mathbf{v}(t')}{|\mathbf{r}-\mathbf{r}'|}\delta[\mathbf{r}'-\mathbf{r}_p(t')]\delta[f(t')], \tag{8.6}$$

where

$$f(t') = t' - t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}: \tag{8.7}$$

The volume integrations can be carried out immediately with the results

$$\Phi(\mathbf{r},t) = \frac{e}{4\pi\varepsilon_0}\int\frac{1}{|\mathbf{r}-\mathbf{r}_p(t')|}\delta[f(t')]dt', \tag{8.8}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 e}{4\pi}\int\frac{\mathbf{v}(t')}{|\mathbf{r}-\mathbf{r}_p(t')|}\delta[f(t')]dt', \tag{8.9}$$

where $f(t_0)$ is now

$$f(t') = t' - t + \frac{|\mathbf{r}-\mathbf{r}_p(t')|}{c}: \tag{8.10}$$

The integral involving the delta function can be simpli ed as

$$\int g(t')\delta[f(t')]dt' = \int g(t')\delta[f(t')]\frac{1}{\left|\frac{df(t')}{dt'}\right|}df = \left.\frac{g(t')}{|df/dt'|}\right|_{f=0}, \tag{8.11}$$

where t_0 is now understood as a solution for t_0 satisfying $f(t_0) = 0$ or

$$t' - t + \frac{|\mathbf{r} - \mathbf{r}_p(t')|}{c} = 0; \tag{8.12}$$

This is in general an implicit equation for t_0: The time derivative of $f(t_0)$ is

$$\frac{df}{dt'} = 1 - \frac{\mathbf{n}(t')\cdot\mathbf{v}_p(t')}{c} = 1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t'), \tag{8.13}$$

where $\mathbf{n}(t_0)$ is the unit vector along the relative distance $\mathbf{r}\ \mathbf{r}_p(t_0)$ and $= \mathbf{v}=c$: The observing time t and time t_0 are related through

$$[1 \quad \mathbf{n}(t_0) \quad (t_0)]dt_0 = dt;$$

or

$$\frac{dt}{dt'} = 1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t');$$

After performing time integration, we nally obtain

$$\Phi(\mathbf{r}, t) = \frac{e}{4\pi\varepsilon_0}\frac{1}{1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t')}\frac{1}{|\mathbf{r} - \mathbf{r}_p(t')|} = \frac{e}{4\pi\varepsilon_0}\frac{1}{\kappa(t')\,|\mathbf{r} - \mathbf{r}_p(t')|}, \tag{8.14}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 e}{4\pi}\frac{1}{1 \quad \mathbf{n}(t_0) \quad (t_0)}\frac{\mathbf{v}(t')}{j\mathbf{r} \quad \mathbf{r}_p(t')j} = \frac{\mu_0 e}{4\pi}\frac{\mathbf{v}(t_0)}{(t_0)j\mathbf{r} \quad \mathbf{r}_p(t_0)j}; \tag{8.15}$$

where

$$(t_0) = 1 \quad \mathbf{n}(t_0) \quad (t_0): \tag{8.16}$$

These retarded potentials, called Lienard-Wiechert potentials, had been formulated in 1898. They are applicable to arbitrary velocity of the charged particle. Retarded nature of the potentials clearly appears in the condition that all time varying quantities, $\mathbf{r}_p(t_0);\mathbf{v}(t_0);\mathbf{n}(t_0);$ must be evaluated at t_0; not at the observing time t because of nite propagation speed of electromagnetic disturbance.

Having found the retarded potentials, we are now ready to calculate the electromagnetic elds due to a moving point charge. The electric eld is to be found from

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{8.17}$$

where spatial and time derivatives pertain to $\mathbf{r}$ (the coordinates of the observing location) and t (observing time). Since the potentials and $\mathbf{A}$ are implicit functions of $\mathbf{r}$ and $t;$ it is more convenient to use the original integral representations, Eqs. (8.8) and (8.9), respectively, for proper di⁄erentiation with respect to $\mathbf{r}$ and $t:$ For example, the spatial derivative of the scalar potential can be performed as follows. Letting $R(t_0)$ = $\mathrm{j}\mathbf{r}\ \mathbf{r}_p(t_0)\mathrm{j};$ and introducing a unit vector in the direction $\mathbf{r}\ \mathbf{r}_p(t_0);$

$$\mathbf{n}(t_0) = \frac{\mathbf{r} \quad \mathbf{r}_p(t_0)}{\mathrm{j}\mathbf{r} \quad \mathbf{r}_p(t_0)\mathrm{j}}; \tag{8.18}$$

we nd

$$\begin{aligned} \nabla\Phi &= \frac{e}{4\pi\varepsilon_0}\int \nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}_p(t')|}\delta[f(t')]\right)dt' \\ &= \frac{e}{4\pi\varepsilon_0}\int \mathbf{n}\frac{\partial}{\partial R}\left(\frac{1}{R}\delta[f(t')]\right)dt' \\ &= -\frac{e}{4\pi\varepsilon_0}\int\left[\frac{\mathbf{n}}{R^2}\delta[f(t')]-\frac{\mathbf{n}}{cR}\frac{d}{df}\delta[f(t')]\right]dt' \\ &= -\frac{e}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa R^2}+\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\mathbf{n}}{\kappa R}\right)\right], \end{aligned} \tag{8.19}$$

where use is made of the integration by parts,

$$\int g(t')\frac{d}{df}\delta[f(t')]dt'$$

$$= \int g(t')\frac{1}{\frac{df}{dt'}}\frac{d}{dt'}\delta[f(t')]dt' \tag{8.20}$$

$$= -\left[\frac{1}{df/dt'}\frac{d}{dt'}\left(\frac{g(t')}{df/dt'}\right)\right]_{f(t')=0} \tag{8.21}$$

$$= -\left[\frac{1}{\kappa(t')}\frac{d}{dt'}\left(\frac{g(t')}{\kappa(t')}\right)\right]_{f(t')=0}. \tag{8.22}$$

Similarly,

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{e}{4\pi\varepsilon_0}\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right), \tag{8.23}$$

and the electric eld becomes

$$\mathbf{E}(\mathbf{r},t) = \frac{e}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa R^2}+\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{f(t')=0}; \tag{8.24}$$

To proceed further, we need concrete expressions for the derivatives,

$$\frac{d\mathbf{n}}{dt_0} \quad \text{and} \quad \frac{d}{dt_0}\left(\frac{1}{R}\right):$$

The unit vector $\mathbf{n}$ along the distance vector $\mathbf{R} = \mathbf{r}$ $\mathbf{r}_p(t_0)$ changes its direction only through the velocity component perpendicular to $\mathbf{R}$;

$$d\mathbf{n} = \frac{\mathbf{v}_?}{R} dt_0; \tag{8.25}$$

as can be seen in Fig. 8-1. This yields

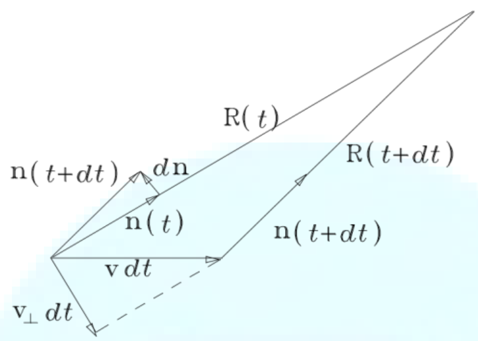


Figure 8-1: The change in the unit vector $\mathbf{n}$ is caused by the perpendicular velocity $\mathbf{v}_?$:

$$\frac{d\mathbf{n}}{dt'} = -\frac{\mathbf{v}_\perp}{R}: \tag{8.26}$$

Also,

$$\frac{d}{dt'}\left(\frac{1}{\kappa R}\right) = -\frac{1}{(\kappa R)^2}\frac{d}{dt'}[(1 \quad \mathbf{n} \quad)R]$$

$$= -\frac{1}{(\kappa R)^2}\left[\mathbf{v}_\perp \quad (\mathbf{n}\,\_)R \quad c(1 \quad \mathbf{n} \quad)\mathbf{n}\ \mathrm{i}\right.$$

$$= -\frac{c}{(\kappa R)^2}\left(\beta^2 \quad \mathbf{n} \quad \frac{1}{c}R(\mathbf{n}\,\_) \quad :\right. \tag{8.27}$$

Substitution of Eqs. (8.26) and (8.27) to Eq. (8.24) gives

$\mathbf{E}(\mathbf{r};t)$ =____ _____ _ ______4"e 0"$\mathbf{n}(R)^2$ $\mathbf{n}(R)^2$ 2 $\mathbf{n}$ $\frac{1}{c}R$($\mathbf{n}$

_) c_____ _2R#

$$= \frac{e}{4\pi\varepsilon_0}\left[\frac{1-\beta^2}{\kappa^3 R^2} \quad (\mathbf{n} \quad) + \frac{1}{c\kappa^3 R}\mathbf{n}\,[(\mathbf{n} \quad) \quad \_]\right. \quad : \qquad (8.28)$$

$f(t0)=0$

The rst term in the RHS,

$$\mathbf{E}_{\text{Coulomb}} = \frac{e}{4\pi\varepsilon_0}\left[\frac{1-\beta^2}{\kappa^3 R^2}(\mathbf{n} \quad)\right._{f(t')=0}, \qquad (8.29)$$

is the Coulomb eld corrected for relativistic e⁄ects. It is proportional to $1=R^2$ and thus does not contribute to radiation of energy. The second term,

$$\mathbf{E}_{\text{rad}} = \frac{e}{4\pi\varepsilon_0 c}\left[\frac{1}{\kappa^3 R}\mathbf{n} \quad [(\mathbf{n} \quad) \quad \_]\right._{f(t')=0}, \qquad (8.30)$$

contains acceleration _ and is proportional to $1=R:$ This is the desired radiation electric eld due to a moving charged particle.

The magnetic eld can be calculated in a similar manner from

$$\mathbf{B} = \quad \mathrm{r} \quad \mathbf{A}$$
$$= \frac{1}{c}[\mathbf{n} \quad \mathbf{E}]_{f(t0)=0}: \qquad (8.31)$$

Derivation of this result is left for an exercise.

8.3 Radiation from a Charge under Linear Acceleration

If the acceleration is parallel (or anti-parallel) to the velocity, _ = 0; the radiation electric eld reduces to

$$\mathbf{E}_{\text{rad}} = \frac{e}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})}{c\kappa^3 R}\right]_{f(t')=0}: \qquad (8.32)$$

The angular distribution of radiation power at the observing time t is

$$\frac{dP(t)}{d\Omega} = c\varepsilon_0 \left|\mathbf{E}_{\text{rad}}\right|^2 R^2$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})}{c\kappa^3}\right]^2_{f(t')=0}: \qquad (8.33)$$

Denoting the angle between _ and $\mathbf{n}$ by ; we have $[\mathbf{n} \quad (\mathbf{n} \quad \_)]^2 = \_^2 \sin^2$; and thus

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\left[\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^6}\right]_{f(t')=0}: \qquad (8.34)$$

However, the power $P(t)$ in the above formulation is the rate of energy radiation at t; the observing time, which is not necessarily equal to the energy loss rate of the charge at the retarded time t_0 determined from $f(t_0) = 0$: To nd the radiation power at the retarded time $P\ (t_0)$; let us consider the amount of di⁄erential energy dE=d radiated during the time interval between t_0 and t_0+dt_0: By de nition dE = $P(t_0)dt_0$: The radiation energy dE is sandwitched between two eccentric spherical surfaces with a volume $dV = R^2cdt_0(1$ cos $)$:

Therefore, the di⁄erential radiation energy is

$$\frac{d\mathcal{E}}{d\Omega} = \frac{1}{2}\varepsilon_0 \left|\mathbf{E}_{\rm rad}\right|^2 R^2 c dt'(1-\beta\cos\theta),$$

and

$$\begin{aligned}\frac{dP(t')}{d\Omega} &= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\left[\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^6}\times(1-\beta\cos\theta)\right]_{f(t')=0} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\left[\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}\right]_{f(t')=0}.\end{aligned} \quad (8.35)$$

In nonrelativistic limit j j 1; the radiation occurs predominantly in the direction perpendicular to the acceleration = =2. The total radiation power in this case is

$$\begin{aligned}P &\simeq \frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{4\pi c}\int\sin^2\theta d\Omega \\ &= \frac{1}{4\pi\varepsilon_0}\frac{2e^2\dot{\beta}^2}{3c}, \quad \beta\ll 1\ :\end{aligned} \quad (8.36)$$

This is the well known Larmor s formula for radiation power due to a nonrelativistic charge. Since in nonrelativistic limit,

n [(**n**) _] ' **n** (**n** _); (8.37)

Larmor s formula is applicable for acceleration in arbitrary direction relative to the velocity.

For arbitrary magnitude of the velocity ; the radiation power can be found from

$$\begin{aligned}P(t') &= \frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{4\pi c}\int\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}d\Omega \\ &= \frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{4\pi c}2\pi\int_0^\pi\frac{\sin^3\theta}{(1-\beta\cos\theta)^5}d\theta \\ &= \frac{1}{4\pi\varepsilon_0}\frac{2e^2\dot{\beta}^2}{3c}\gamma^6,\end{aligned} \quad (8.38)$$

where

1

(8.39) p =;

1 2

is the relativity factor and the following integral is used,

$$\int_0^{\pi} \frac{\sin^3\theta}{(1-\beta\cos\theta)^5}d\theta = \int_1^1 \frac{1-x^2}{(1-\beta x)^5}dx = \frac{4}{3}\frac{1}{(1-\beta^2)^3}; \quad (8.40)$$

The angular dependence of the radiation intensity,

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^6}, \quad (8.41)$$

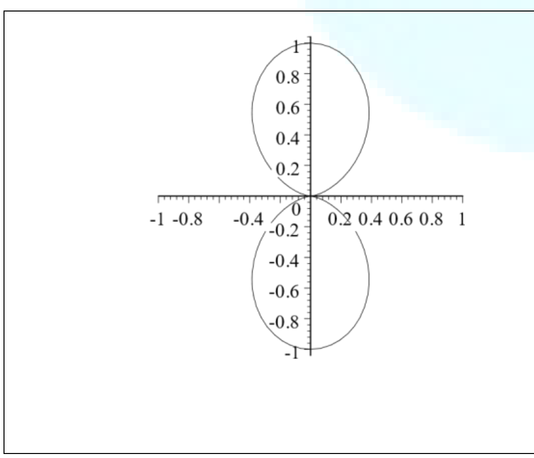
peaks at angle $_0$ where

$$\cos\theta_0 = \frac{\sqrt{1+24\beta^2}-1}{4\beta}: \quad (8.42)$$

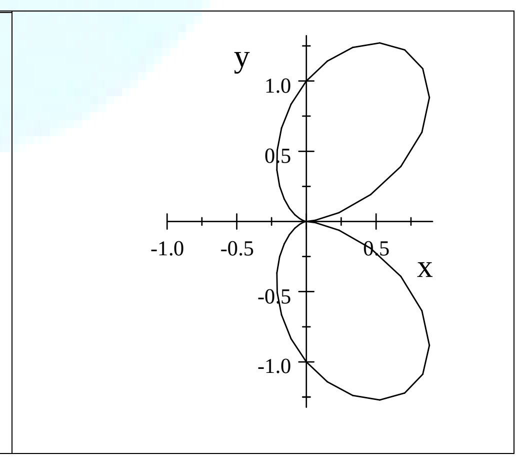
In highly relativistic limit 1; the angle $_0$ becomes of order

$$_0 ' \frac{1}{\_} 1; \quad (8.43)$$

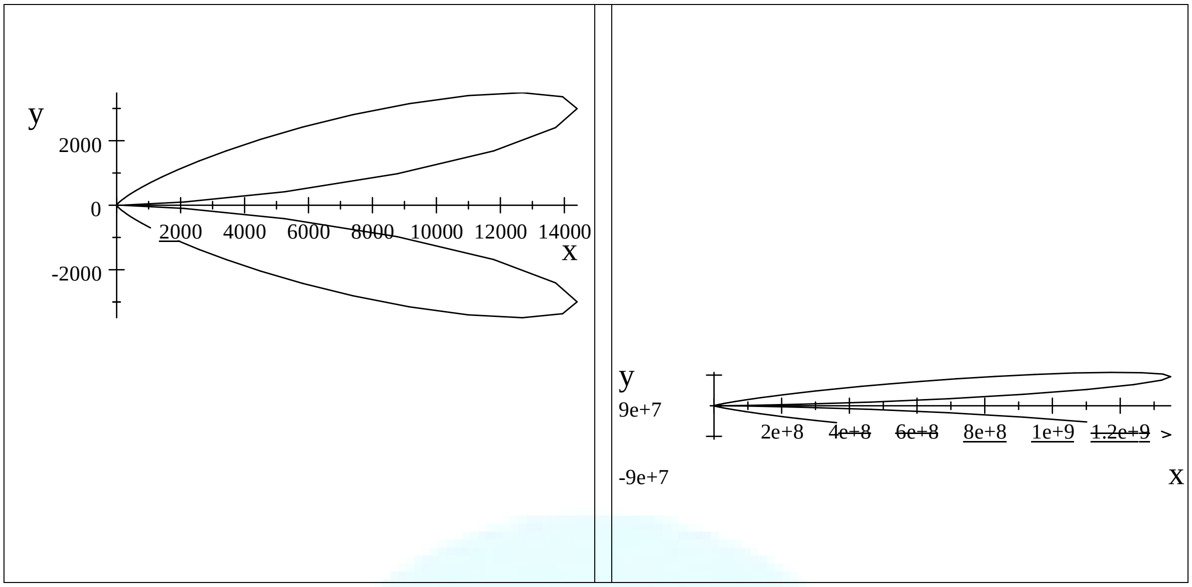
which indicates a very sharp pencil or beam of radiation along the direction of the velocity : (This is also the case for acceleration perpendicular to the velocity as shown in the following section.) Angular distribution of radiation intensity I ( )) for = 0; 0:2; 0:9 and 0:999 is shown below for a common acceleration. Note that the radiation intensity rapidly increases with = 1=p1 2;



= 0 (= 1:0):



= 0:2 (= 1:02):



= 0:9 (= 2:3): = 0:99 (= 7:09):

Polar plot of I( ) for several factors but with common parallel acceleration.

For linear acceleration, the momentum change of the charged particle is

$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{mv}{\sqrt{1-\beta^2}}\right) = m\gamma^3\dot{v}. \tag{8.44}$$

Therefore, the radiation power can be rewritten as

$$\begin{aligned} P(t') &= \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3m^2}\left(\frac{dp}{dt'}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{2e^4}{3c^3m^2}E_{ac}^2 \\ &= \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3m^2}\left(\frac{d\mathcal{E}}{dx'}\right)^2, \end{aligned} \tag{8.45}$$

where E_{ac} is an external acceleration electric eld and

$$\frac{d\mathcal{E}}{dx'},$$

is the energy gradient of a linear accelerator which is at most of the order of 100 MeV/m in practice. The radiation loss in linear accelerators is negligibly small compared with energy gain. This is one of the advantages of high energy linear accelerators. Note that the radiation power due to linear acceleration is independent of the particle energy or the relativity factor :

8.4 Radiation from a Charge in Circular Motion

In circular motion, the acceleration is perpendicular to the velocity. Here we consider highly relativistic motion of a charged particle with ’ 1, for nonrelativistic case has already been discussed in Chapter 4. A geometry convenient for analysis to follow is shown in Fig.8-2. A particle undergoes circular motion with an orbit radius in the x z plane and it passes the origin at $t_0 = 0$: At that instant, the acceleration is in the x direction while the velocity is in the z direction,

$$\_ = \_ \mathbf{e}_x; \qquad = \mathbf{e}_z:$$

The radiation electric eld can then be written down in terms of cartesian components,

$$\mathbf{E}(\mathbf{r},t) = \frac{e}{4\pi\varepsilon_0}\frac{1}{c\kappa^3 R}\mathbf{n}\times[(\mathbf{n} \quad) \quad \_]$$

$$= \frac{e}{4\pi\varepsilon_0}\frac{|\dot{\boldsymbol{\beta}}|}{c\kappa^3 R}[(\beta-\cos\theta)\cos \quad \mathbf{e} + (1 \qquad \cos\ )\sin\ \mathbf{e}\ ]; \qquad (8.46)$$

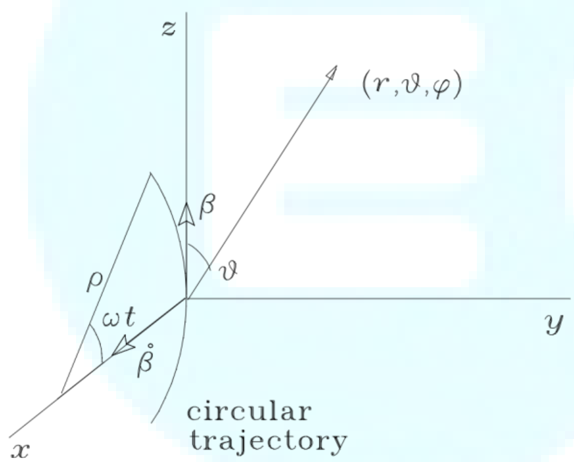


Figure 8-2: Particle undergoing circular motion in the x z plane with radius and frequency $!_0$: At $t = 0$; the particle passes the origin. and the angular distribution of radiation power is given by

$$\frac{dP(t_0)}{d} = \frac{1}{4''_0}\frac{e^2\_^{\,2}}{4c}\frac{1}{(1 \qquad \cos\ )^5} \qquad \cos\theta)^2 - \frac{1}{\gamma^2}\sin^2\theta\cos^2 \qquad (1:$$

(8.47)

The total radiation power is

$$
\begin{aligned}
P(t') &= \frac{1}{4\pi\varepsilon_0}\frac{e^2\left|\dot{\boldsymbol{\beta}}\right|^2}{4\pi c}\int\frac{1}{(1-\beta\cos\theta)^5}\left[(1-\beta \quad \theta \quad - \gamma \quad \cos\theta)^2-\frac{1}{\gamma^2}\sin^2\theta\cos^2\phi\right]d\Omega \\
&= \frac{1}{4\pi\varepsilon_0}\frac{e^2\left|\dot{\boldsymbol{\beta}}\right|^2}{4\pi c}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\phi\frac{1}{(1-\beta\cos\theta)^5}\left[(1 \quad \cos)^2 \quad \frac{1}{\gamma^2}\sin^2 \quad \cos^2\right. \\
&= \frac{1}{4\pi\varepsilon_0}\frac{2e^2|\dot{\mathbf{v}}|^2}{3c^3}\gamma^4.
\end{aligned}
\tag{8.48}
$$

Relevant integrals are:

$$
\int_{-1}^1\frac{dx}{(1-\beta x)^3}=\frac{2}{(1-\beta^2)^2}=2\gamma^4,
$$

$$
\int_{-1}^1\frac{1-x^2}{(1-\beta x)^5}dx=\frac{4}{3}\frac{1}{(1-\beta^2)^3}=\frac{4}{3}\gamma^6:
$$

In highly relativistic case, the acceleration may be approximated by

$$
|\dot{\mathbf{v}}|=\frac{v^2}{\rho}\simeq\frac{c^2}{\rho}: \tag{8.49}
$$

Then, the radiation power in terms of the orbit radius is

$$
P(t')\simeq\frac{1}{4\pi\varepsilon_0}\frac{2e^2c\gamma^4}{3\rho^2}: \tag{8.50}
$$

To maintain the radiation loss in a circular accelerator at a tolerable level, the orbit radius must be increased as the particle energy mc^2 increases. Note that the radiation power is a sensitive function of the particle energy in contrast to the case of linear acceleration.

As an example, let us consider the Betatron, the well known inductive electron accelerator invented by Kerst. In the Betatron, the electron cyclotron orbit is maintained constant,

$$
\frac{mc^2}{} = ecB\,(t) = ecB_0\sin !t\,'\,ecB_0!t;
$$

where only the initial phase of the sinusoidal magnetic eld is useful for acceleration, $!t$ 1. Then the rate of electron energy gain is

$$
\frac{d}{dt}\left(\gamma mc^2\right)=ec\rho B_0\omega=\text{const.}
$$

Equating this to the radiation energy loss,

$$
P\left(t'\right)=\frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{c^2}{\rho}\right)^2\gamma^4,
$$

we nd

$$_{\max} \simeq \left(\frac{3\omega e B_0 \rho^3}{2mc^2 r_e}\right)^{1/4}, \tag{8.51}$$

where

$$r_e = \frac{e^2}{4\pi\varepsilon_0 mc^2} = 2.8 \times 10^{-15} \text{ m}, \tag{8.52}$$

is the classical radius of electron. If = 50 cm, ! = 2 60 rad/s, and B_0 = 0:5 T (5 kG), the upper limit of is about 400 and the maximum electron energy attainable is approximately 200 MeV.

8.5 Fourier Spectrum of Radiation Fields

The formulae such as Eqs. (8.45) and (8.50) only tell us the total radiation power integrated over the frequency. Radiation eld emitted by highly relativistic particle is hardly monochromatic but cnsists of broad frequency spectrum. Knowing such frequency spectrum of radiation is of practical importance for identifying radiation source. A typical example is synchrotron radiation due to highly relativistic electrons bent by, or trapped in, a magnetic eld. In nonrelativistic limit, the radiation elds all have a single frequency component corresponding to the classical electron cyclotron frequency $!_c$ = eB=m: However, as the relativity factor increases, the radiation elds consist of harmonics of the fundamental frequency $!_c$ = eB= m. In highly relativistic case 1; the frequency spectrum becomes almost continuous peaking at the frequency ! ' ${}^3!_c$ = ${}^2eB$=m:

Frequency spectrum of the radiation electric eld can be formulated by directly applying Fourier transformation on the eld in Eq. (8.24),

$$\begin{aligned}(\mathbf{r},\omega) &= \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)e^{i\omega t}dt \qquad \mathbf{E} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{e}{c}\int_{-\infty}^{\infty}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\kappa^3 R}\right]_{f(t')=0} e^{i\omega t}dt.\end{aligned} \tag{8.53}$$

Changing the variable from t to t_0 and assuming r r_p; we obtain

$\mathbf{E}(\mathbf{r};!)$ = 4"1 0 ec $e^{i!r=cr}$ Z11 $\mathbf{n}$ [($\mathbf{n}$ 2)] exp $i!$ t^0 1$_c\mathbf{n}$ $\mathbf{r}_p(t_0)$ dt_0: (8.54)

The unit vector $\mathbf{n}(t_0)$ may be regarded constant since r r_p and thus approximated by $\mathbf{n}$ ' $\mathbf{r}$=r:

Then, $\mathbf{n}$ [($\mathbf{n}$) _] d $\mathbf{n}$ ($\mathbf{n}$)

2 ' dt_0 (; (8.55)

and Eq. (8.53) can be integrated by parts,

$$\mathbf{E}(\mathbf{r},\omega) = -\frac{i\omega}{4\pi\varepsilon_0}\frac{e}{c}\frac{e^{i\omega r/c}}{r}\int_{-\infty}^{\infty}\mathbf{n}\quad(\mathbf{n}\quad)\exp i\omega\, t' \quad \tfrac{1}{c}\mathbf{n}\,\mathbf{r}_p(t') \quad dt' : \tag{8.56}$$

Any time derivatives contained in the physical electric eld are merely multiplied by $i\omega$ in the Fourier space and the disappearance of the acceleration _ is not surprising. Amazing fact about Eq. (8.56) is that it is applicable to radiation elds which do not require particle acceleration such as Cherenkov and transition radiation provided a proper velocity of electromagnetic waves in dielectrics is substituted for c:

The radiation energy (not power) associated with the electric eld is

$$c\varepsilon_0 r^2 \int\int |\mathbf{E}(\mathbf{r},t)|^2\, dt\, d\Omega \ \text{(J)}. \tag{8.57}$$

However, since the electric eld $\mathbf{E}(\mathbf{r},t)$ and its Fourier transform $\mathbf{E}(\mathbf{r},\omega)$ are related through

Parseval s theorem:

$$\int |\mathbf{E}(\mathbf{r},t)|^2\, dt = \frac{1}{2\pi}\int |\mathbf{E}(\mathbf{r},\omega)|^2\, d\omega, \tag{8.58}$$

the radiation energy can be written in terms of the Fourier transform $\mathbf{E}(\mathbf{r},\omega)$ as

$$\frac{1}{2\pi}c\varepsilon_0 r^2 \int\int |\mathbf{E}(\mathbf{r},\omega)|^2\, d\Omega\, d\omega. \tag{8.59}$$

The quantity

$$\frac{dI(\omega)}{d\Omega} \equiv \frac{1}{2\pi} c\varepsilon_0 r^2 \left|\mathbf{E}(\mathbf{r},\omega)\right|^2, \tag{8.60}$$

can therefore be identi ed as the radiation energy per unit solid angle per unit frequency. Substituting Eq. (8.56), we nd

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{8\pi^2 c}\omega^2 \left|\int \mathbf{n} \quad (\mathbf{n} \quad) \exp\left[i\omega\left(t' - \frac{1}{c} \mathbf{n} \quad \mathbf{r}_p(t')\right)\right] dt'\right|^2, \quad -\infty < \omega < \infty, \tag{8.61}$$

or

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi^2 c}\omega^2 \left|\int \mathbf{n} \quad (\mathbf{n} \quad) \exp\left[i\omega\left(t' - \frac{1}{c}\mathbf{n}\cdot\mathbf{r}_p(t')\right)\right] dt'\right|^2, \ 0 < \omega < \infty : \tag{8.62}$$

Let us work on a few examples.

8.6 Synchrotron Radiation I

Synchrotron radiation is due to highly relativistic electrons trapped in a magnetic eld. The radiation beam rotates together with an electron and is directed along the direction of the velocity with an angular spread of order ' 1= 1: If the orbiting frequency is !$_0$ = eB= m_e; the radiation beam shines a detector for a duration

$$\Delta t' \simeq \frac{\Delta\theta}{\omega_0} = \frac{1}{\gamma\omega_0}, \tag{8.63}$$

as seen by the electron at the retarded time t_0. Since

$$\frac{\Delta t}{\Delta t'} = 1 \quad \mathbf{n} \quad = 1 \quad \simeq \frac{1}{2\gamma^2}, \tag{8.64}$$

which is entirely due to Doppler e⁄ect, the pulse width detected is of order

$$\Delta t \simeq \frac{\Delta t'}{2\gamma^2} = \frac{1}{2\gamma^3\omega_0}: \tag{8.65}$$

Therefore, synchrotron radiation is dominated by frequency components in the range

$$! \simeq \gamma^3\omega_0 = \gamma^2\frac{eB}{m_e}: \tag{8.66}$$

In ultrarelativistic case, the frequency spectrum of synchrotron radiation can extend to very high frequencies even for a modest magnetic eld.

We calculate the amount of energy radiated in one period of cyclotron motion T = 2=!$_0$: Since the radiation power is constant, the radiated energy is

$$\mathcal{E} = \frac{1}{T}\int_0^\infty I(\omega)\,d\omega = \frac{\omega_0}{2\pi}\int_0^\infty I(\omega)\,d\omega.$$

13 (8.67)

The time integration in the energy spectrum,

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi^2 c}\omega^2\left|\int_{-T/2}^{T/2} \mathbf{n} \quad (\mathbf{n} \quad)\exp\left[i\omega\left(t' - \frac{1}{c}\ \mathbf{n}\ \mathbf{r}_p(t')\right)\right]dt'\right|^2,\ 0<\omega<\infty, \quad (8.68)$$

can be extended from 1 to 1 since the characteristic frequency of the radiation eld is much higher than the

fundamental frequency *! !*$_0$*;*

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi^2 c}\omega^2\left|\int_{-\infty}^{\infty} \mathbf{n} \quad (\mathbf{n} \quad)\exp\left[i\omega\left(t' - \frac{1}{c}\mathbf{n}\ \mathbf{r}_p(t0)\right)\ dt0\right]\right|^2 ;\ 0 < ! < 1:$$

(8.69)

To perform the integration, we assume the trajectory shown in Fig. (8-2) in which an electron passes the origin at t = 0*:* The vector $\mathbf{n}$ is assumed to be in the y z plane since the radiation pro le is essentially symmetric about the z axis. The trajectory is described by

$$\mathbf{r}_p\, t0 = \quad 1 \quad \cos !_0 t0\, \mathbf{e}_x + \sin !_0 t \mathbf{e}_z; \quad (8.70)$$

and the velocity is

$$= \frac{\omega_0\rho}{c}\left(\sin\omega_0 t\mathbf{e}_x + \cos\omega_0 t\mathbf{e}_z\right): \quad (8.71)$$

Then,

$$\frac{1}{c}\mathbf{n}\cdot\mathbf{r}_p(t') = \frac{\rho}{c}\sin(\omega_0 t)\cos : \quad (8.72)$$

Since

$$\mathbf{n} \quad (\mathbf{n} \quad) = \quad ? = \quad \sin !_0 t\mathbf{e}_x + \cos !_0 t \sin\ \mathbf{e}?; \quad (8.73)$$

where

$$\mathbf{e}? = \mathbf{n} \quad \mathbf{e}_x; \quad (8.74)$$

is a unit vector perpendicular to both $\mathbf{n}$ and x axis, and the radiation lasts for a very short time and is limited within a small angle *;* Eq. (8.73) reduces to

$$?' \quad !_0 t\mathbf{e}_x + \mathbf{e}?: \quad (8.75)$$

Within the same order of accuracy, the phase function $!(t \quad \mathbf{n}\ \mathbf{r}_p = c)$ can be approximated by

! t n r^p = ! t

sin!0tcos c

c

$$\simeq \frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t+\frac{c^2}{3\rho^2}t^3\right], \tag{8.76}$$

where v has been approximated by c but 1 by

$$1 \simeq \frac{1}{2\gamma^2}:$$

Then,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\boldsymbol{\beta}_\perp \exp\left[i\omega\left(t-\frac{\mathbf{n}\cdot\mathbf{r}_p}{c}\right)\right]dt\\
=&\int_{-\infty}^{\infty}(\omega_0 t\mathbf{e}_x-\theta\mathbf{e}_\perp)\exp\left\{\frac{i\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t+\frac{c^2}{3\rho^2}t^3\right]\right\}dt\\
=&\ 2i\omega_0\int_0^{\infty}t\sin\left\{\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t+\frac{c^2}{3\rho^2}t^3\right]\right\}dt\ \mathbf{e}_x\\
&-2\theta\int_0^{\infty}\cos\left\{\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t+\frac{c^2}{3\rho^2}t^3\right]\right\}dt\ \mathbf{e}_\perp.
\end{aligned} \tag{8.77}$$

The integrals reduce to the modi ed Bessel functions of fractional orders (or the Airy s functions),

$$\int_0^{\infty}\cos\left[\frac{3}{2}x\left(\tau+\frac{1}{3}\tau^3\right)\right]d\tau=\frac{1}{\sqrt{3}}K_{1/3}(x), \tag{8.78}$$

$$\int_0^{\infty}\tau\sin\left[\frac{3}{2}x\left(\tau+\frac{1}{3}\tau^3\right)\right]d\tau=\frac{1}{\sqrt{3}}K_{2/3}(x), \tag{8.79}$$

where

$$\tau=\frac{\gamma\omega_0 t}{\sqrt{1+\gamma^2\theta^2}},\ x=\frac{\omega}{3\gamma^3\omega_0}\left(1+\gamma^2\theta^2\right)^{3/2}. \tag{8.80}$$

Then

$$2i\omega_0\int_0^{\infty}t\sin\left\{\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t+\frac{c^2}{3\rho^2}t^3\right]\right\}dt=2i\frac{\frac{1}{\gamma^2}+\theta^2}{\omega_0}\frac{1}{\sqrt{3}}K_{2/3}(x),$$

$$2\theta\int_0^{\infty}\cos\left\{\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t+\frac{c^2}{3\rho^2}t^3\right]\right\}dt=2\theta\frac{\sqrt{\frac{1}{\gamma^2}+\theta^2}}{\omega_0}\frac{1}{\sqrt{3}}K_{2/3}(x),$$

and for I (!)=d ; we obtain

$$\begin{aligned}
\frac{dI(\omega)}{d\Omega}&=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{3\pi^2 c}\frac{\omega^2}{\omega_0^2}\left(\tfrac{1}{\gamma^2}+\theta^2\right)\left[\left(\tfrac{1}{\gamma^2}+\theta^2\right)K_{2/3}^2(x)+\theta^2K_{1/3}^2(x)\right]\\
&=\frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma^2}{\pi^2 c}\hat{\omega}^2\left(1+\Theta^2\right)\left\{\left(1+\Theta^2\right)K_{2/3}^2[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]+\Theta^2K_{1/3}^2[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]\right\},
\end{aligned}$$

where = and !^ is the normalized frequency,

$$\hat{\omega} = \frac{\omega}{3\gamma^3\omega_0}. \tag{8.81}$$

The modi ed Bessel functions $K_{2/3}(x)$ and $K_{1/3}(x)$ both diverge at $x \to 0$. However, $xK_{2/3}(x)$ and $xK_{1/3}(x)$ are well behaving and vanish at small x. Fig. (8-3) shows $2x^2K^2_{2/3}(x)$ and $x^2K^2_{1/3}(x)$ which represent radiation intensities associated with electric eld polarization along $\mathbf{e}_x$ (that is, in the particle orbit plane) and $\mathbf{e}_\perp$; respectively.

The energy spectrum $I(\omega)$ emitted during one revolution ($T = 2\pi/\omega_0$) can be found by inte-

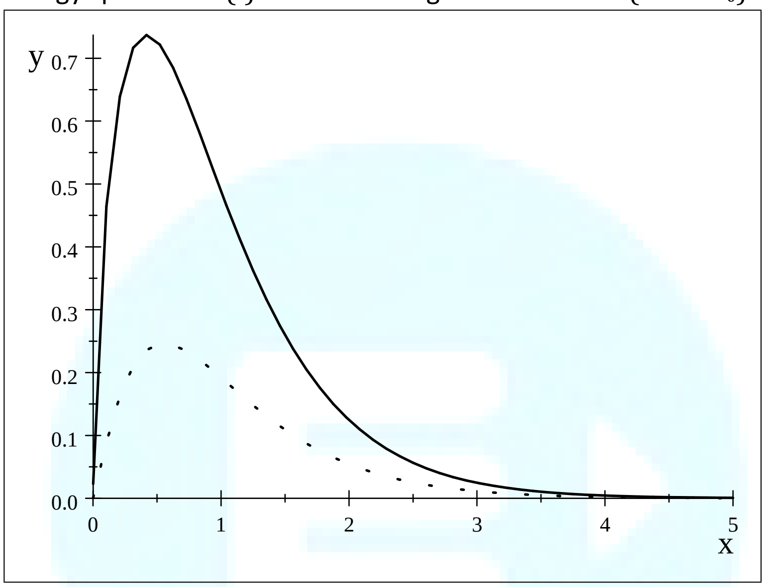


Figure 8-3: $2x^2K^2_{2/3}(x)$ (solid line) and $x^2K^2_{1/3}(x)$ (dotted line). The factor of 2 in $2x^2K^2_{2/3}(x)$ assumes ’ 1= :

grating $dI(\omega)/d\Omega$ over the solid angle,

$$\begin{aligned}
& I(\omega) \\
&= \frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma^2}{\pi^2 c}\hat{\omega}^2\int_0^{\pi}\sin\theta' d\theta'\int_0^{2\pi}d\phi'\left(1+\Theta^2\right) \\
&\quad \times\left\{\left(1+\Theta^2\right)K^2_{2/3}[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]+\Theta^2K^2_{1/3}[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]\right\} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{6e^2\gamma^2}{\pi c}\hat{\omega}^2\int_{-\pi/2}^{\pi/2}\cos\theta d\theta\left(1+\Theta^2\right)\left\{\left(1+\Theta^2\right)K^2_{2/3}[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]+\Theta^2K^2_{1/3}[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]\right\} \\
&\simeq \frac{1}{4\pi\varepsilon_0}\frac{6e^2\gamma}{\pi c}\hat{\omega}^2\int_{-\infty}^{\infty}\left(1+\Theta^2\right)\left\{\left(1+\Theta^2\right)K^2_{2/3}[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]+\Theta^2K^2_{1/3}[\hat{\omega}\left(1+\Theta^2\right)^{3/2}]\right\}d\Theta \\
&= \frac{1}{4\pi\varepsilon_0}\frac{6e^2\gamma}{\pi c}f(\hat{\omega}),
\end{aligned} \tag{8.82}$$

where $0 = \_2$ is the polar angle from the axis of electron revolution (y axis), 0 is the azimuthal angle about the y axis, and the function $f(!\hat{})$ is de ned by

$$f(\hat{\omega}) = 2\hat{\omega}^2 \int_0^{\infty} \left(1+\Theta^2\right) \left\{ \left(1+\Theta^2\right) K_{2/3}^2[\hat{\omega}\left(1+\Theta^2\right)^{3/2}] + \Theta^2 K_{1/3}^2[\hat{\omega}\left(1+\Theta^2\right)^{3/2}] \right\} d\Theta, \quad (8.83)$$

and shown in Fig. (8-4) (linear scale) and Fig. (8-5) (log-log scale). In Fig. (8-5), the straight line in the low frequency regime $! \ {}^3!_0$ has a slope of $1{=}3;$ and indicates $I(!) \_ !^{1=3}: f(!\hat{})$ peaks at $!\hat{} \ ' \ 0{:}14;$ or $! \ ' \ 0{:}42 \ {}^3!_0;$ and its peak value is about 0.83. In high frequency regime $! \ {}^3!_0;$ the spectrum decays exponentially. The energy radiated per revolution can be calculated as

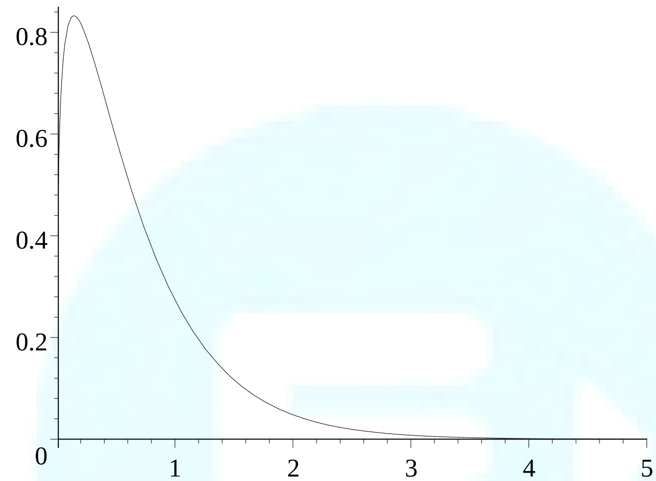


Figure 8-4: The function $f(x) = f \ !{=}3 \ {}^3!_0$ plotted in linear scale.

$$\mathcal{E} = \int_0^{\infty} I(\omega)\, d\omega \simeq \frac{1}{4\pi\varepsilon_0} \frac{6e^2\gamma}{\pi c} 3\gamma^3\omega_0 \int_0^{\infty} f(\hat{\omega})\, d\hat{\omega}; \quad (8.84)$$

where the integral numerically evaluated is approximately

$$\int_0^{\infty} f(\hat{\omega})\, d\hat{\omega} \simeq 0.713 :$$

Then

$$\mathcal{E} = \int_0^{\infty} I(\omega)\, d\omega = \frac{1}{4\pi\varepsilon_0} \frac{4.08e^2\gamma^4\omega_0}{c}, \ \text{(J)}$$

and the radiation power is

$$P = \mathcal{E}\frac{\omega_0}{2\pi} = \frac{1}{4\pi\varepsilon_0} \frac{0.65e^2 c\gamma^4}{\rho^2}, \quad (8.85)$$

which agrees reasonably well with Eq. (8.50),

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2e^2 c\gamma^4}{3\rho^2} \simeq \frac{1}{4\pi\varepsilon_0} \frac{0.667e^2 c\gamma^4}{\rho^2}, \ \text{(W)}.$$

The discrepancy may be attributed to the various approximations made in the analysis.

8.7 Synchrotron Radiation II

An alternative approach to nding the frequency spectrum of synchrotron radiation is to apply discrete Fourier analysis in terms of harmonics of the fundamental frequency $!_0$ directly to radiation

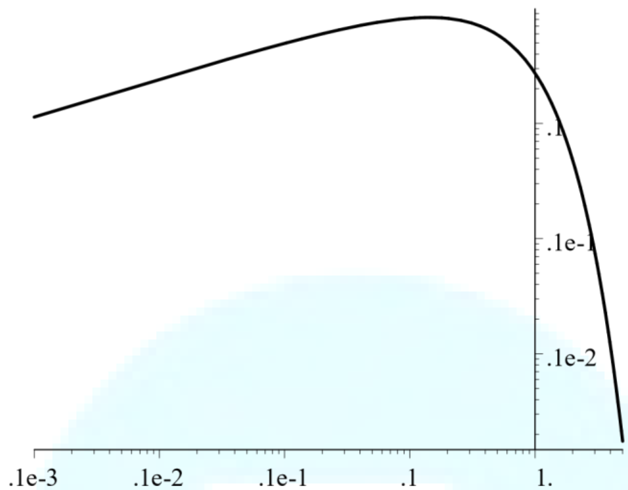


Figure 8-5: Log-log plot of the function $f(x) = f\,!{=}3\ {}^3!_0:$

elds. If a charge e is in circular motion with a constant angular frequency $!_0$, the radiation eld contains higher harmonics of $!_0$ and can be Fourier decomposed as follows. Let us recall the Lienard-Wiechert vector potential,

$$\mathbf{A}(\mathbf{r};t) = \frac{{}_0 \qquad e\mathbf{v}(t_0)}{4 \quad (1 \quad \mathbf{n} \quad)R(t_0)}\Big|_{f(t_0)=0} \quad ; \qquad (8.86)$$

where $R = \mathrm{j}\mathbf{r}\ \mathbf{r}_p(t_0)\mathrm{j}$ and the subscript $f(t_0) = 0$ indicates that all time dependent quantities should be evaluated at the retarded time t_0 determined from the implicit equation for t_0,

$$f(t_0) = t_0 \quad t + \mathrm{j} \quad \frac{\ \ p\ \ |}{c} = 0 \ : \qquad (8.87)$$

The vector potential can be Fourier decomposed as

$$\mathbf{r} \quad \mathbf{r}\,(t_0)$$

$$\mathbf{A}(\mathbf{r},t) = \sum_l \mathbf{A}_l(\mathbf{r})e^{-il\omega_0 t}, \tag{8.88}$$

where

$$\mathbf{A}_l(\mathbf{r}) = \frac{\mu_0 e}{4\pi}\frac{1}{T}\int_0^T \frac{e\mathbf{v}(t')}{(1-\mathbf{n}\cdot\boldsymbol{\beta})R}e^{il\omega_0 t}dt, \tag{8.89}$$

with $T = 2{=}!_0$ being the period of the circular motion. Changing the integration variable from t to $t0$ by noting

$$\frac{dt}{dt'} = 1 \qquad \mathbf{n} \qquad ; \tag{8.90}$$

leads to

$$\mathbf{A}_l(\mathbf{r}) \simeq \frac{\mu_0 e}{4\pi r}\frac{e^{ik_l r}}{T}\int_0^T \mathbf{v}(t')e^{i(l\omega_0 t' - k_l \mathbf{n}\cdot\mathbf{r}_p)}dt', \tag{8.91}$$

where $k_l = l!_0{=}c$. Note that the period T remains unchanged through the transformation. Let the particle trajectory be

$\mathbf{r}_p(t0) = (\cos !_0 t0\mathbf{e}_x + \sin !_0 t0\mathbf{e}_y);$ (8.92) $\mathbf{v}(t0) = !_0($ $\sin !_0 t0\mathbf{e}_x +$

$\cos !_0 t0\mathbf{e}_y):$ (8.93)

Since all radiation elds rotate with the charge, the observing point can be chosen at arbitrary azimuthal angle and we choose = =2; so that $\mathbf{n} = (1; ; = =2):$ Then

$$\mathbf{n}\ \mathbf{r}_p(t0) = \quad \sin\sin !_0 t0: \tag{8.94}$$

The velocity in the spherical coordinates is

$$\mathbf{v}(t0) = !_0(\sin\cos !_0 t0\mathbf{e}_r + \cos \quad \cos !_0 t0\mathbf{e} + \sin !_0 t0\mathbf{e}\ ): \tag{8.95}$$

Thus, the component of $\mathbf{A}_l(\mathbf{r})$ is given by

$$A_{l\theta} = \frac{\mu_0 e}{4\pi r}e^{ik_l r}\frac{\rho\omega_0}{T}\cos\theta\int_0^T \cos\omega_0 t' e^{il\omega_0(t'-\rho\sin\theta\sin\omega_0 t'/c)}dt' : \tag{8.96}$$

Letting $x = \omega_0 t'$, and noting $= !_0{=}c;$ we can rewrite this as

$$A_{l\theta} = e^{ik_l r}\frac{\mu_0 e\rho\omega_0}{4\pi r}\frac{\cos\theta}{2\pi}\int_0^{2\pi}\cos x e^{il(x \quad \beta\sin\theta\sin x)}dx. \tag{8.97}$$

The integral reduces to

$$\begin{aligned} &\int_0^{2\pi}\cos x e^{il(x-\beta\sin\theta\sin x)}dx \\ &= \pi[J_{l+1}(l\beta\sin\theta) + J_{l \quad 1}(l\beta\sin\theta)] \\ &= \frac{2\pi}{\beta\sin\theta}J_l(l\beta\sin\theta), \end{aligned}$$

and thus nally,

$$A_{l\theta} = \frac{\mu_0 ec}{4\pi r}e^{ik_l r}\cot\theta J_l(l\beta\sin\theta) : \tag{8.98}$$

Similarly,

$$A_{l\phi} = i\frac{\mu_0 e\rho\omega_0}{4\pi r} e^{ik_l r} J_l'(l\beta\sin\theta), \tag{8.99}$$

where use has been made of the recurrence formula of the Bessel functions,

$$J_{l\,1}(x) \qquad J_{l+1}(x) = 2J_{l0}(x): \tag{8.100}$$

The far- eld radiation magnetic eld can be found from

$$\mathbf{H}_l \simeq \frac{i}{\mu_0}\mathbf{k}_l \times \mathbf{A}_l, \tag{8.101}$$

which yields

$$H_{l\theta} = \frac{e\rho\omega_0 k_l}{4\pi r} e^{ik_l r} J_l'(l\beta\sin\theta), \tag{8.102}$$

$$H_{l\phi} = i\frac{eck_l}{4\pi r} e^{ik_l r} \cot J_l(l \sin): \tag{8.103}$$

The radiation power associated with the l-th harmonic is

$$\begin{aligned} P_l &= c\mu_0 r^2 \int |H_l|^2 d\Omega \\ &= \frac{1}{8\pi\epsilon_0}\frac{(el\omega_0)^2}{c}\int_0^\pi \left[\beta^2 J_l'^2(l\beta\sin\theta) + \cot^2\theta J_l^2(l\beta\sin\theta)\right]\sin\theta d\theta. \end{aligned} \tag{8.104}$$

Since $P_l = P_{\ l}$, the total power is

$$P = \sum_{l=1}^{\infty} P_l, \tag{8.105}$$

where P_l is now

$$P_l = \frac{1}{4\pi\epsilon_0}\frac{(el\omega_0)^2}{c}\int_0^\pi \left[\beta^2 J_l'^2(l\beta\sin\theta) + \cot^2\theta J_l^2(l\beta\sin\theta)\right]\sin\theta d\theta, \quad l \geq 1: \tag{8.106}$$

In nonrelativistic limit 1; the l = 1 term is dominant. For x 1;

$$J_1(x) \simeq \frac{1}{2}x, \ \ J_1'(x) \simeq \frac{1}{2}: \tag{8.107}$$

Then the lowest order radiation power agrees with the Larmor s formula,

$$\begin{aligned} P_1 &\simeq \frac{1}{4\pi\epsilon_0}\frac{(e\omega_0)^2}{c}\frac{\beta^2}{4}\int_0^\pi (1+\cos^2\theta)\sin\theta d\theta \\ &= \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3}, \end{aligned} \tag{8.108}$$

where $a = v^2 =$ is the acceleration.

The integral in Eq. (8.105) cannot be reduced to elementary functions. However, the total power given in Eq. (8.105) should reduce to Eq. (8.50),

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3}\gamma^4 = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3}\frac{1}{(1-\beta^2)^2}: \tag{8.109}$$

To show this, we modify the integral by noting

$$J_l^2(x) = \frac{1}{\pi}\int_0^\pi J_0(2x\sin\theta)\cos(2l)d\theta, \tag{8.110}$$

$$J_l'^2(x) = \frac{1}{\pi}\int_0^\pi J_0(2x\sin\theta)\left(\cos 2\theta - \frac{l^2}{x^2}\right)\cos(2l)d\theta. \tag{8.111}$$

Then

$$^2J_l'^2(\beta l\sin\theta) + \cot^2\theta J_l^2(\beta l\sin\theta) = \frac{1}{\pi}\int_0^\pi J_0(2\beta l\sin\theta\sin x)(\beta^2\cos 2x - 1)\cos(2lx)dx. \tag{8.112}$$

Furthermore,

$$\int_0^\pi J_0(2\beta l\sin x\sin\theta)\sin\theta d\theta = \frac{\sin(2\beta l\sin x)}{\beta l\sin x}, \tag{8.113}$$

and the power P_l reduces to

$$P_l = \frac{1}{4\pi\epsilon_0}\frac{(e\omega_0 l)^2}{\pi c}\int_0^\pi \frac{\sin(2\beta l\sin x)}{\beta l\sin x}(\beta^2\cos 2x - 1)\cos(2lx)dx. \tag{8.114}$$

Noting

$$\begin{aligned}
&\int_0^\pi \sin x\sin(2\beta l\sin x)\cos(2lx)dx \\
=\ & \frac{1}{2}\int_0^\pi [\sin(2l+1)x + \sin(1-2l)x]\sin(2\beta l\sin x)dx \\
=\ & \frac{\pi}{2}[J_{2l+1}(2\beta lx) - J_{2l-1}(2\beta l)] \\
=\ & -\pi J_{2l}'(2\beta l)
\end{aligned} \tag{8.115}$$

and

$$\begin{aligned}
&\int_0^\pi \frac{\sin(2\beta l\sin x)\cos(2lx)}{\sin x}dx \\
=\ & 2l\int_0^\pi\int_0^\beta \cos(2l\beta\sin x)\cos(2lx)dxd\beta \\
=\ & 2\pi l\int_0^\beta J_{2l}(2l\beta)d\beta,
\end{aligned} \tag{8.116}$$

the power P_l can be rewritten as

$$P_l = \frac{1}{4\pi\epsilon_0}\frac{2(e\omega_0 l)^2}{c\beta l}\left[\beta^2 J_{2l}'(2l\beta) - l(1-\beta^2)\int_0^\beta J_{2l}(2l\beta)d\beta\right], \tag{8.117}$$

and the total power is

$$P = \sum_{l=1}^\infty P_l\ ;$$

Relevant sum formula of the Bessel functions is

$$\sum_{l=1}^\infty J_{2l}(2lx) = \frac{x^2}{2(1-x^2)}; \tag{8.118}$$

Di⁄erentiating by *x;*

$$\sum_l 2lJ'_{2l}(2lx) = \frac{x}{(1-x^2)^2}: \tag{8.119}$$

Also,

$$\sum_l l^2 \int_0^\beta J_{2l}(2lx)dx = \frac{\beta^3}{6(1-\beta^2)^3}: \tag{8.120}$$

This is one of Kapteyn series formulae. (See for example, *Mathematical Formulae* (in Japanese), (Iwanami, Tokyo, 1960), vol. 3, p. 212.) Then, nally, the total radiation power becomes

$$P = \frac{1}{4\pi\epsilon_0}\frac{2e^2a^2}{3c^3}\gamma^4, \quad a = \frac{v^2}{\rho} \tag{8.121}$$

which is consistent with the known radiation power from a charge undergoing circular motion.

Analytic expression for the radiation power P_l can be found by exploiting following approximation,

$$J_{2l}(2\beta l) \simeq \frac{1}{\sqrt{\pi}l^{1/3}} Ai\left(\frac{l^{2/3}}{\gamma^2}\right), \tag{8.122}$$

where Ai(x) is the Airy function de ned by

$$Ai(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right)dt.$$

$$f(x) = \frac{1}{\sqrt{\pi}}\int_0^{20} \cos\left(\frac{1}{3}t^3 + xt\right)dt \tag{8.123}$$

For large x 1; the function takes the form

$$Ai(x) \simeq \frac{1}{2x^{1/4}}\exp\left(-\frac{2}{3}x^{3/2}\right), \ x \gg 1: \tag{8.124}$$

Also, $Ai_0(0)$ = 0:4587: Using these approximations, we nd the following approximate formulae,

$$P_l \simeq \frac{1}{4\pi\varepsilon_0} \times 0.5175 \frac{e^2\omega_0^2}{c} l^{1/3}, \ 1 \ll l \ll \gamma^3, \tag{8.125}$$

$$P_l \simeq \frac{1}{4\pi\varepsilon_0}\frac{e^2\omega_0^2}{2\sqrt{\pi}c}\sqrt{\frac{l}{\gamma}}\exp\left(-\frac{2}{3}\frac{l}{\gamma^3}\right), \ l \gg \gamma^3: \tag{8.126}$$

Note that the radiation power increases with l in the manner P_l _ $l^{1=3}$ up to l ' 3 beyond which P_l decays exponentially. This is consistent with the analysis in the preceding section.

8.8 Free Electron Laser

In a synchrotron radiation source, many beamlines can be installed by bending an electron beam. In straight sections of race track, no radiation occurs. However, by inserting a device called wiggler,

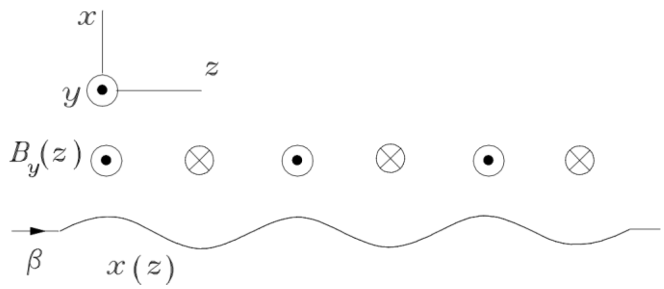


Figure 8-6: In a wiggler, an electron beam is modulated by a periodic magnetic eld. Electrons acquire spatially oscillating perpendicular displacement $x(z)$ and velocity $v_x(z)$ which together with the radiation magnetic eld B_{Ry} produces a ponderomotive force $\mathbf{v}_x(z)$ $\mathbf{B}_{Ry}(z)$ directed in the z direction. The force acts to cause electron bunching required for ampli cation of coherent radiation.

high intensity radiation can be extracted. A wiggler consists of periodically alternating magnets and gives an electron beam periodic kick perpendicular to both the beam velocity and magnetic eld. Electrons receive kicks at an interval

$$\Delta t' = \frac{\lambda_w}{c}, \tag{8.127}$$

where $_w$ is the wavelength of the periodic wiggler structure. Because of Doppler shift, this time

1 ' 2 for a stationary

detector in front of the beam, interval is shortened by a factor 2

1

$$\Delta t \simeq \frac{\lambda_w}{2\gamma^2 c}. \tag{8.128}$$

Therefore, the wavelength of resultant radiation is approximately given by

$$\simeq \frac{\lambda_w}{2\gamma^2} \ll \lambda_w, \tag{8.129}$$

and the frequency by

$$! \simeq \frac{4\pi\gamma^2 c}{\lambda_w} : \tag{8.130}$$

The intensity of free electron laser can be orders of magnitude higher than that of synchrotron radiation because of coherent ampli cation through the periodic structure. Electrons tend to be bunched in the wiggler as the electron beam travels through the periodic structure. In contrast, no collective interaction between electrons and electromagnetic waves exists in synchrotron radiation. In electron bunching, the magnetic ponderomotive force plays a major role. Let us assume a periodic wiggler magnetic eld in y direction,

$$B_y = B_0 \cos\left(\frac{2\pi}{\lambda_w} z\right) = B_0 \cos k_w z, \;\; k_w = \frac{2\pi}{\lambda_w}: \quad (8.131)$$

The Lorentz force is

$$\mathbf{F} = e\mathbf{v} \quad \mathbf{B};$$

or

$$F_x = \quad ev_z B_0 \cos k_w z:$$

Then electron acquires a velocity v_x in x direction,

$$v_x = \frac{eB_0}{m\gamma k_w} \sin k_w z, \quad (8.132)$$

and a resultant ponderomotive force is

$$e\mathbf{v}_x \quad \mathbf{B}_R; \quad (8.133)$$

where $\mathbf{B}_R$ is the radiation magnetic eld propagating in the form $e^{i(kz \; !t)}$ along the beam. Since the acceleration due to the wiggler magnetic eld is in x direction, the radiation electric eld is predominantly in x direction and radiation magnetic eld B_R is in y direction. Then the ponderomotive force directed in z direction is proportional to $e^{i[(k+k_w)z \; !t]}$ and propagates at a velocity

$$\frac{\omega}{k + k_w}:$$

When this propagation velocity matches the electron beam velocity c, strong interaction between the radiation eld and electron motion takes place and electrons tend to be bunched. This results in positive feedback for wave ampli cation. From the condition

$$\frac{\omega}{k + k_w} \simeq \beta c, \text{ or } \frac{k}{k + k_w} = \beta,$$

we readily recover

$$k = \frac{\quad}{1} k_w \,'\, 2\;{}^2 k_w:$$

8.9 Radiation Accompanying Decay

Equation (8.61) for the angular distribution of radiation energy can be applied to cases in which particle acceleration is not involved explicitly. In decay, an energetic electron (or positron) is suddenly released from a nucleus together with neutrino. The situation is equivalent to sudden acceleration of an electron. The duration of acceleration t is limited by the uncertainty principle $t \; mc^2$ &~: Therefore, the upper limit of the frequency spectrum should be of the order of

$$!^{\max} \simeq \frac{\gamma mc^2}{\hbar}: \quad (8.134)$$

In decay, the maximum value of is of order of 30.

Let us assume that an electron suddenly acquires a velocity = $v{=}c$ and then travels at a constant velocity. The integration in Eq. (8.61) is limited from $t = 0$ to $1;$

$$\begin{aligned}\frac{dI(\omega)}{d\Omega} &= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi^2 c}\omega^2\beta^2\sin^2\theta\left|\int_0^\infty e^{i\omega(1-\beta\cos\theta)t}dt\right|^2\\ &= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi^2 c}\frac{\beta^2\sin^2\theta}{(1-\beta\cos\theta)^2},\end{aligned} \tag{8.135}$$

where is the angle between the velocity and the unit vector $\mathbf{n}$: Integration over the solid angle yields 2

$$I(\omega) = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\pi c}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right.\right. \quad 2 = \text{const.} \lesssim \omega_{\max} = \frac{\gamma mc}{\hbar}; \; ! \; : \tag{8.136}$$

The frequency spectrum is at up to $!_{\max}$: Therefore, the total energy radiated through decay is approximately given by

$$\mathcal{E} \simeq \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\pi c}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right. \quad 2\right]\frac{\gamma mc^2}{\hbar} \simeq \frac{1}{\pi}\left[\ln(4\gamma^2)-2\right]\alpha\gamma mc^2, \tag{8.137}$$

where the dimensionless quantity ,

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{c\hbar} \simeq \frac{1}{137}, \tag{8.138}$$

is the ne structure constant. The energy emitted as radiation through decay is a small fraction of the electron energy.

8.10 Cherenkov Radiation

Cherenkov radiation occurs when a charged particle travels faster than electromagnetic waves in a material medium. It does not require acceleration of charges and the basic mechanism is very similar to that of sound shock waves in gases. As in the case of decay, we assume a charge travelling along a straight line at a velocity = $v{=}c(!)$; where

$$c(\omega) = \frac{1}{\sqrt{\varepsilon(\omega)\mu_0}},$$

is the velocity of electromagnetic waves in a dielectric having a permittivity $"(!)$: Eq. (8.61) should be modi ed as follows after taking into account the proper de nition of $c(!)$;

$$\frac{dI(\omega)}{d\Omega} = \frac{\mu_0}{4\pi}\frac{e^2}{4\pi^2 c(\omega)}\omega^2 v^2\sin^2\theta\left|\int_{-\infty}^{\infty} e^{i\omega(1\quad\beta\cos\theta)t}dt\right|^2 : \tag{8.139}$$

Note that the integration limits are from 1 to 1: The time integral is singular,

$$\int_{-\infty}^{\infty} e^{i\omega(1-\beta\cos\theta)t}dt = 2\pi\delta[\omega(1 \quad \cos)];$$

and the condition for radiation is

$$\cos \quad = \frac{1}{\ } < 1;$$

or

$$(\omega) = \frac{v}{c(\omega)} > 1 \, ; \tag{8.140}$$

Then,

$$\frac{dI(\omega)}{d\Omega} = \frac{\mu_0}{4\pi}\frac{e^2}{(2\pi)^2 c(\omega)}\omega^2 v^2 \sin^2\theta\,\delta^2[\omega(1 \quad \cos)]: \tag{8.141}$$

Square of a delta function is not integrable and the radiation energy simply diverges. This is merely due to the assumption that the charge is radiating forever from t = 1 to 1 which is of course unphysical. It is more appropriate to consider a radiation *power* rather than energy. For this purpose, we consider a thin slab of the dielectric of thickness dz. The transit time over the distance dz is T = dz=v and we calculate energy radiated during that time,

$$\frac{dI(\omega)}{d\Omega} = \frac{\mu_0}{4\pi}\frac{e^2}{4\pi^2 c(\omega)}\omega^2 v^2 \sin^2\theta \left| \int_{-\Delta T/2}^{\Delta T/2} e^{i\omega(1-\beta\cos\theta)t}dt \right|^2 : \tag{8.142}$$

The integral can be carried out easily,

$$\int_{\Delta T/2}^{\Delta T/2} e^{i\omega(1-\beta\cos\theta)t}dt = \frac{2}{\omega(1-\beta\cos\theta)}\sin\left[(1 \quad \cos\theta)\frac{\Delta T\omega}{2}\right]:$$

Thus

$$\frac{dI(\omega)}{d\Omega} = \frac{\mu_0}{4\pi}\frac{e^2\omega^2}{4\pi^2 c(\omega)}\sin^2\theta\left(\frac{\sin\alpha}{\alpha}\right)^2 (dz)^2, \tag{8.143}$$

where

$$= \frac{1}{2}(1 \quad \cos)! \quad T: \tag{8.144}$$

In high frequency regime ! T 1; the function (sin=)^2 may be approximated by a delta function ( ): Integration over the solid angle yields

$$\begin{aligned} -\frac{dI(\omega)}{dz} &= \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{c_0}\right)^2 \omega\left(1-\frac{1}{\beta^2}\right) \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{c_0}\right)^2 \omega\left(1-\frac{1}{\varepsilon(\omega)\mu_0 v^2}\right), \end{aligned} \tag{8.145}$$

where c_0 = 1=p"$_{0\,0}$ is the speed of light in vacuum. The rate of energy loss due to Cherenkov emission is given by

$$-\frac{d\mathcal{E}}{dz} = \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{c_0}\right)^2 \int_0^\infty \omega\left(1 - \frac{1}{\varepsilon(\omega)\mu_0 v^2}\right)d\omega, \quad 0 < \omega < \infty \; : \tag{8.146}$$

The result obtained is meaningful only if

$$> 1; \quad \text{or} \quad v > c(\omega) = \frac{1}{\sqrt{\varepsilon(\omega)\mu_0}}, \tag{8.147}$$

which is the condition for Cherenkov radiation. The instantaneous radiation power can be estimated from

$$\begin{aligned} P &= \frac{d}{dt}\int I(\omega)d\omega \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{c_0}\right)^2 v \int \omega\left(1 - \frac{1}{\varepsilon(\omega)\mu_0 v^2}\right)d\omega. \end{aligned} \tag{8.148}$$

In order to nd the eld pro les emitted through Cherenkov radiation, we start from the wave equations for the potentials in a material medium,

$$\left(\nabla^2 - \mu_0\tilde{\varepsilon}\frac{\partial^2}{\partial t^2}\right)\tilde{\varepsilon}\Phi(\mathbf{r}, t) = \quad {}_{\text{free}}; \tag{8.149}$$

$$\left(\nabla^2 - \mu_0\tilde{\varepsilon}\frac{\partial^2}{\partial t^2} \quad \mathbf{A}(\mathbf{r}; t) = \quad {}_0\mathbf{J}; \tag{8.150}$$

where ~" is the dielectric operator containing time derivative,

$$\tilde{\varepsilon} = \tilde{\varepsilon}\left(\frac{\partial}{\partial t} \quad : \tag{8.151}$$

For a charged particle e travelling at a constant velocity $\mathbf{v}$; the charge density and current density are described by

$$= e\,(\mathbf{r} \quad \mathbf{v}t); \tag{8.152}$$

$$\mathbf{J} = e\mathbf{v}\,(\mathbf{r} \quad \mathbf{v}t): \tag{8.153}$$

Then, after Fourier-Laplace transformation, the Fourier potentials can readily be found,

$$\Phi(\mathbf{k}, \omega) = \frac{2\pi e}{\varepsilon(\omega)\left(k^2 - \frac{\omega^2}{c^2(\omega)}\right)}\delta(\quad ! \quad \mathbf{k}\,\mathbf{v}); \tag{8.154}$$

$$\mathbf{A}(\mathbf{k}, \omega) = \frac{2\pi e\mu_0\mathbf{v}}{k^2 - \frac{\omega^2}{c^2(\omega)}}\delta(\quad ! \quad \mathbf{k}\,\mathbf{v}); \tag{8.155}$$

where, as before,

$$c^2(\omega) = \frac{1}{\varepsilon(\omega)\mu_0}, \tag{8.156}$$

and the transformation

$$\int\int \delta(\mathbf{r} - \mathbf{v}t) e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} dV\, dt = 2\pi\delta(\omega - \mathbf{k}\cdot\mathbf{v}); \qquad (8.157)$$

is substituted. Since the physical electric field is

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t), \qquad (8.158)$$

the Fourier component of the electric field is given by

$$\begin{aligned} \mathbf{E}(\mathbf{k}, \omega) &= -i\mathbf{k}\Phi(\mathbf{k}, \omega) + i\omega\mathbf{A}(\mathbf{k}, \omega) \\ &= -2\pi i e \frac{\left(\mathbf{k} - \frac{\omega}{c^2}\mathbf{v}\right)}{\varepsilon(\omega)\left(k^2 - \dfrac{\omega^2}{c^2(\omega)}\right)} \delta(\omega - \mathbf{k}\cdot\mathbf{v}). \end{aligned} \qquad (8.159)$$

Similarly, the Fourier-Laplace component of the magnetic field is

$$\begin{aligned} \mathbf{B}(\mathbf{k}, \omega) &= i\mathbf{k} \times \mathbf{A}(\mathbf{k}, \omega) \\ &= 2\pi i e \mu_0 \frac{\mathbf{k} \times \mathbf{v}}{k^2 - \dfrac{\omega^2}{c^2(\omega)}} \delta(\omega - \mathbf{k}\cdot\mathbf{v}). \end{aligned} \qquad (8.160)$$

The physical electromagnetic fields can then be found through inverse transformations,

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega\, \mathbf{E}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}; \qquad (8.161)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega\, \mathbf{B}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \qquad (8.162)$$

To proceed further, we assume that the charged particle is travelling along the z axis at a constant velocity $\mathbf{v}$. The system is symmetric about the axis and we may assume an observing point in the $x\,z$ plane without loss of generality. We denote the cylindrical coordinates of the

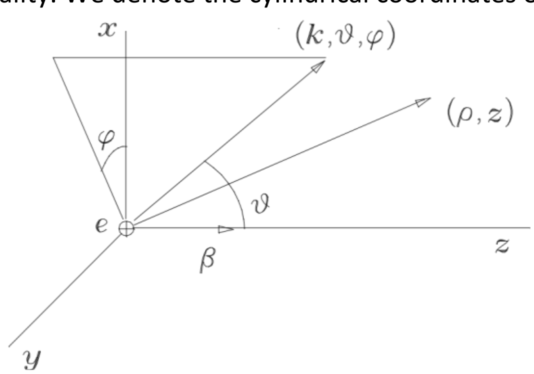


Figure 8-7: Geometry for Fourier inverse transform.

observing point by $(; = 0; z)$ and spherical Fourier coordinates by $\mathbf{k} = (k; ;):$ Then,

$$\mathbf{k}\ \mathbf{r} = k(z \cos \quad + \quad \sin \cos):$$

Because of the cylindrical symmetry, radiation of energy is expected in the radial direction ; and the relevant Poynting vector is

$$\begin{aligned} S &= (\mathbf{E} \quad \mathbf{H}\ ) \\ &= \quad E_z(\mathbf{r};t)H\ (\mathbf{r};t): \end{aligned} \tag{8.163}$$

The energy radiated per unit length along the particle trajectory (z axis) can be calculated from

$$\begin{aligned} -\frac{d\mathcal{E}}{dz} &= -2\pi\rho \int E_z(\mathbf{r},t)H_\phi^*(\mathbf{r},t)dt \\ &\quad -\rho \int E_z(\mathbf{r},\omega)H_\phi^*(\mathbf{r},\omega)d\omega, = \end{aligned} \tag{8.164}$$

where $E_z(\mathbf{r};!)$ is the Laplace transform of the electric eld,

$$E_z(\mathbf{r};!) = \frac{1}{(2\ )^3} Z\ d^3kE_z(\mathbf{k};!)e^{i\mathbf{k}\mathbf{r}}; \tag{8.165}$$

and $H\ (\mathbf{r};!)$ is the Laplace transform of the magnetic eld,

$$H_\phi(\mathbf{r},\omega) = \frac{1}{(2\pi)^3}\int d^3kH_\phi(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{r}}: \tag{8.166}$$

The Laplace transform of the axial electric eld $E_z(\mathbf{r};!)$ can be calculated as follows:

$$\begin{aligned} E_z(\mathbf{r},\omega) &= \frac{1}{(2\pi)^3}\int d^3k E_z(k,\omega)e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= -\frac{ie}{(2\pi)^2}\int_0^\infty k^2dk\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi\frac{k\cos\theta-\frac{\omega v}{c^2(\omega)}}{\varepsilon(\omega)\left(k^2-\frac{\omega^2}{c^2(\omega)}\right)}\delta(\omega-kv\cos\theta)e^{ik(z\cos\theta+\rho\sin\theta\cos\phi)} \\ &= -\frac{i\mu_0 e\omega}{(2\pi)^2}\int_{1/\beta}^\infty \kappa d\kappa\frac{\frac{1}{\beta^2}-1}{\kappa^2-1}\int_0^{2\pi}\exp\left[\frac{i\omega}{c(\omega)}\left\{\frac{z}{\beta}+\kappa\rho\sqrt{1-\frac{1}{(\beta\kappa)^2}}\cos\phi\right\}\right]d\phi \\ &= -\frac{i\mu_0 e\omega}{2\pi}\exp\left(\frac{i\omega z}{\beta c(\omega)}\right)\int_{1/\beta}^\infty \kappa d\kappa\frac{\frac{1}{\beta^2}-1}{\kappa^2-1}J_0\left(\frac{\omega\rho}{c(\omega)}\sqrt{\kappa^2-\frac{1}{\beta^2}}\right), \end{aligned} \quad (8.167)$$

where

$$= \frac{kc(\omega)}{\omega},\ \beta = \frac{v}{c(\omega)}; \quad (8.168)$$

Letting

$$^2 = \kappa^2 - \frac{1}{\beta^2}, \quad (8.169)$$

we nally obtain

$$\begin{aligned} E_z(\mathbf{r},\omega) &= \frac{i\mu_0 e\omega}{2\pi}\exp\left(\frac{i\omega z}{\beta c(\omega)}\right)\left(1-\frac{1}{\beta^2}\right)\int_0^\infty \frac{\zeta J_0\left(\frac{\omega\rho\zeta}{c(\omega)}\right)}{\zeta^2-1+\frac{1}{\beta^2}}d\zeta \\ &= \frac{ie\mu_0}{2\pi}\omega\exp\left(\frac{i\omega z}{\beta c(\omega)}\right)\left(1-\frac{1}{\beta^2}\right)K_0\ (\qquad);\end{aligned}$$

(8.170)

where

$$\alpha = \pm\frac{\omega}{c(\omega)}\sqrt{\frac{1}{\beta^2}-1}, \quad (8.171)$$

and use is made of the integral representation of the modi ed Bessel function $K_0(ax)$;

$$K_0(ax) = \int_0^\infty \frac{tJ_0(at)}{t^2+x^2}dt. \quad (8.172)$$

In the asymptotic regime j j 1; $K_0($ $)$ approaches

r $\frac{\pi}{}$ e :

(8.173)

2

For this to be propagating radially outward in the form e^{ik} ; we must choose

$$= -\frac{\omega}{c(\omega)}\sqrt{\frac{1}{\beta^2}-1}$$

$$= -i\frac{\omega}{c(\omega)}\sqrt{1-\frac{1}{\beta^2}}, \quad \beta > 1$$

30 : (8.174)

With this choice for ; the Laplace transform of the axial electric eld becomes proportional to

$$\exp\left[\frac{i\omega}{c(\omega)}\left(\sqrt{1-\frac{1}{\beta^2}}\rho+\frac{z}{\beta}\right)\right], \tag{8.175}$$

and the radial and axial wavenumbers can be identi ed as

$$k_\rho = \frac{\omega}{c(\omega)}\sqrt{1-\frac{1}{\beta^2}}, \quad k_z = \frac{\omega}{v}, \tag{8.176}$$

respectively. Cherenkov radiation is con ned in a cone characterized by an angle ;

$$\sin\theta = \frac{c(\omega)}{v}, \tag{8.177}$$

as shown in Fig. 8-8.

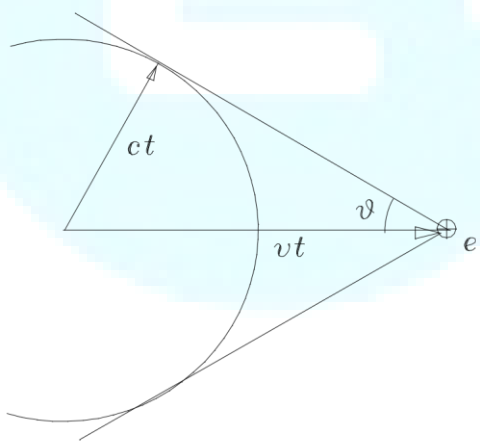


Figure 8-8: Cherenkov cone. Radiation elds are con ned in the cone.

The Laplace transform of the azimuthal magnetic eld is given by

$$
\begin{aligned}
H_\phi(\mathbf{r},\omega) &= \frac{1}{(2\pi)^3}\int d^3k H_\phi(k,\omega)e^{ik\cdot r} \\
&= -\frac{ie}{(2\pi)^2}\int k^2dk\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi \frac{kv\sin\theta}{k^2-\frac{\omega^2}{c^2(\omega)}}\delta(\omega-kv\cos\theta) \\
&\quad \times e^{ik(z\cos\theta+\rho\sin\theta\cos\phi)} \\
&= \frac{e}{2\pi}\alpha K_1(\alpha\rho)\exp\left(\frac{i\omega}{c(\omega)}z\right),
\end{aligned}
\tag{8.178}
$$

where the following integral

$$
\int_0^\infty \frac{x^2J_1(bx)}{x^2+a^2}dx = aK_1(ab),
\tag{8.179}
$$

is noted. (Calculation steps are left for an exercise.) Substituting $E_z(r;!)$ and H $(r;!)$ into Eq. (8.164), we nd

$$
-\frac{d\mathcal{E}}{dz} = -\frac{i\mu_0e^2}{(2\pi)^2}\rho\int_{-\infty}^\infty \omega\left(1-\frac{1}{\beta^2}\right)\alpha^*K_0(\alpha\rho)K_1^*(\alpha\rho)d\omega.
\tag{8.180}
$$

In the asymptotic region j j 1; this reduces to

$$
-\frac{d\mathcal{E}}{dz} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{c_0^2}\int_0^\infty \omega\left(1-\frac{1}{\beta^2}\right)d\omega,
\tag{8.181}
$$

in agreement with the earlier result, Eq. (8.146). Eq. (8.180) can be used even when Cherenkov condition is not satis ed. In this case, energy loss is through near eld Coulomb interaction between a charged particle and ions and electrons in molecules in the dielectric media. We will return to this problem in Section 8.12.

8.11 Transition Radiation

Transition radiation occurs when a charge crosses a boundary of two dielectric media. No acceleration is required, nor is it necessary for charge to move faster than the speed of light as in Cherenkov radiation. In this respect, transition radiation is a least demanding radiation mechanism. Radiation emitted from a charge approaching a conductor is an extreme case of transition radiation with an in nite permittivity, and may be regarded as the inverse process of radiation accompanying decay. Disappearance, rather than creation, of charge is responsible for transition radiation.

We rst consider a simple case: a charge e approaching normally a conducting plate at a velocity v (> 0). On impact, the charge is assumed to come to rest. A conducting plate is mathematically equivalent to an in nitely permissive dielectric plate. An image charge e moving in the opposite direction in the conducting plate can be introduced so that the current density is

$J_z(\mathbf{r};t)$ = ev [$(z + vt)$ + $(z$ $vt)$] (x) (y); 1 < t < 0; (8.182)

where at $t = 0$ (or $z = 0$) the particle is brought to rest. Its Laplace transform is

$$
\begin{aligned}
J_z(\mathbf{r},\omega) &= \int_{-\infty}^0 J_z(\mathbf{r},t)e^{i\omega t}dt \\
&= -e\exp\left(-i\frac{\omega}{v}|z|\right)\delta(x)\delta(y):
\end{aligned}
\tag{8.183}
$$

Let the observing point be at P located at $(r;)$: (The system is symmetric about the axis and thus

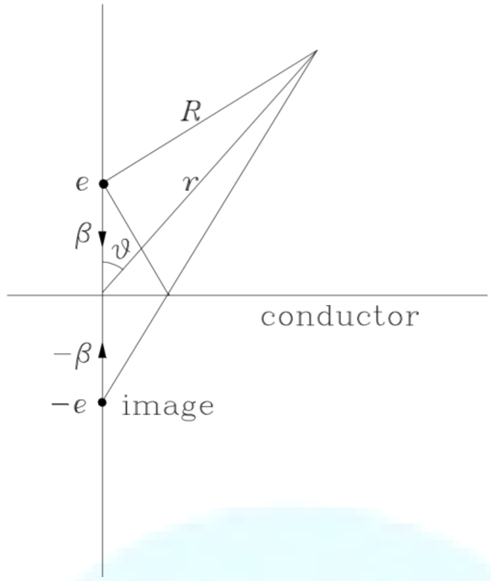


Figure 8-9: Radiation from a charge impinging on a metal surface. Sudden deceleration at the metal surface is the inverse process of radiation accompanying beta decay.

is ignorable.) The vector potential at P can be calculated in the usual manner,

$$= -\frac{\mu}{4\pi r} e \int_0 \left[\exp\left(-i\frac{}{v} z\right) e \qquad \theta + \exp\left(-i\frac{}{v} z\right) e^{ikz'\cos\theta}\right] dz'$$

$$= i\frac{\mu_0 e\beta}{4\pi r} e^{ikr} \frac{1}{k}\left(\frac{1}{1+\beta\cos\theta} + \frac{1}{1-\beta\cos\theta}\right), \quad z > 0, \qquad (8.184)$$

$$A_z(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int \frac{J_z(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|} dV' \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int J_z(\mathbf{r}',\omega) e^{-i\mathbf{k}\cdot}$$

$$\_\ \mu_0 e \quad ikr \quad \infty \quad \left(- \ \omega \ '\right) \quad ikz'\cos\theta \quad \left(- \ \omega \ ' \ \mathbf{r}_0 dV\, 0\right.$$

and the magnetic eld from

$$H_\phi(\mathbf{r},\omega) \simeq -ikA_z \sin\theta/\mu_0$$
$$= \frac{e\beta}{4\pi r} e^{ikr}\left(\frac{1}{1+\beta\cos\theta} + \frac{1}{1-\beta\cos\theta}\right)\sin : \qquad (8.185)$$

The angular distribution of radiation energy is thus given by

$$\begin{aligned}\frac{dI(\omega)}{d\Omega} &= \frac{1}{2\pi}r^2 c\mu_0 \left|H_\phi(\mathbf{r},\omega)\right|^2 \\ &= \frac{e^2\beta^2 c\mu_0}{32\pi^3}\left(\frac{1}{1+\beta\cos\theta}+\frac{1}{1-\beta\cos\theta}\right)^2 \sin^2\theta \\ &= \frac{1}{4\pi\varepsilon_0}\frac{e^2 v^2}{2\pi^2 c^3}\frac{\sin^2\theta}{(1-\beta^2\cos^2\theta)^2}, \quad -\infty < \omega < \infty\end{aligned} \tag{8.186}$$

Integration over the solid angle in the region $z > 0$ (0 < < =2) yields

$$I(\omega) = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2\pi c}\frac{1}{\beta}\left(\left(1+\beta^2\right)\ln\frac{1+\beta}{1-\beta} - 2\beta\right), \ \omega > 0$$

Evidently, the radiation energy $\int_0^\infty I(\omega)d\omega$ diverges. This is due to the assumption of a perfect conductor. In practice, metals cannot be regarded as perfect conductor. The condition that the surface impedance of metal

$$Z = \sqrt{\frac{-i\omega\mu_0}{-i\omega\varepsilon_0+\sigma}},$$

be su¢ ciently small compared with the free space impedance Z_0 = p 0="0 imposes an upper limit of the frequency, !"0 and a cuto⁄ emerges in the integral $\int_0^{\omega_{max}} I(\omega)d\omega$.

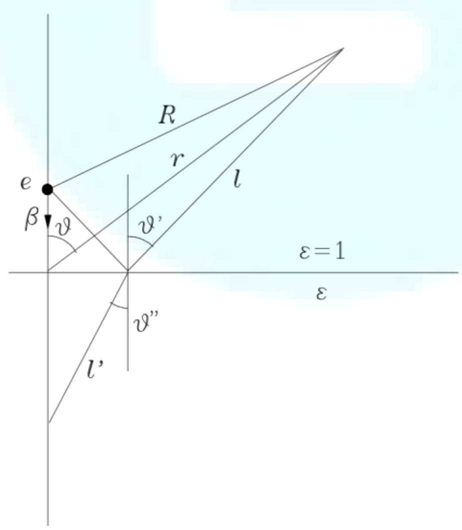


Figure 8-10: Transition radiation emitted by a charge passing through a dielectric boundary.

We now analyze the case of a dielectric slab having a relative permittivity "r = "="0: In this case, the particle continues to travel after passing the boundary and the current density is now

$J_z(\mathbf{r};t) =$ ev (x) (y) $(z + vt);$ $1 < t < 1:$ (8.187)

Its Laplace transform is

$$J_z(\mathbf{r},\omega) = -ev\delta(x)\delta(y)e^{\ i\frac{\omega}{v}t}: \tag{8.188}$$

The contribution to the radiation elds from the region $z > 0$ consists of two parts, one directly from the charge (as in free space) and the other via re ection at the dielectric boundary. Denoting the magnetic re ection coe¢ cient by in the Fresnel s formulae,

$$\Gamma = \frac{\varepsilon_r \cos\theta - \sqrt{\varepsilon_r - \sin^2\theta}}{\varepsilon_r \cos\theta + \sqrt{\varepsilon_r - \sin^2\theta}}, \tag{8.189}$$

and following the same procedure as in the case of conductor plate, we nd

$$H_{\phi 1}(\mathbf{r},\omega) \simeq \frac{ev}{4\pi cr}e^{ikr}\left(\frac{1}{1+\beta\cos\theta} + \frac{\Gamma}{1-\beta\cos\theta}\right)\sin\theta, \quad z > 0 \ : \tag{8.190}$$

The contribution from the region $z < 0$ involves refraction at the boundary, and thus additional retardation because of the longer path length. In Fig.8-10, for $z < 0$, we observe

$$l = r + z'\tan\theta''\sin\theta,\ z' < 0$$
$$l' = -z'/\cos\theta'',$$
$$\frac{\sin\theta'}{\sin\theta''} = \sqrt{\varepsilon},$$

and thus

kl + pkl_ 0 = kr kzp sin 2: (8.191)

Then the contribution from the region $z < 0$ to the integral becomes

$$H_{\phi 2}(\mathbf{r},\omega) = -\frac{ev}{4\pi cr}(1-\Gamma)\frac{1}{1+\beta\sqrt{\varepsilon_r - \sin^2\theta}}e^{ikr}\sin \tag{8.192}$$

Note that the factor 1 here indicates the eld amplitude transmitted into the air region. The total magnetic eld is H 1 + H 2; and the angular distribution of the radiation energy is given by

$$\frac{dI(\omega,\theta)}{d\Omega} = \frac{r^2 c\mu_0}{2\pi}|H_\phi(\mathbf{r},\omega)|^2 = \frac{\mu_0 e^2 v^2}{32\pi^3 c}A^2\sin^2\theta, \tag{8.193}$$

where

$$A = \frac{1}{1+\beta\cos\theta} + \frac{\Gamma}{1-\beta\cos\theta} - \frac{1-\Gamma}{1+\beta\sqrt{\varepsilon_r - \sin^2\theta}}: \tag{8.194}$$

In the case of ideal conductor "$_r$! 1; = 1; we recover

$$\frac{dI(\omega,\theta)}{d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{e^2v^2}{2\pi^2c^3}\left(\frac{\sin\theta}{1-\beta^2\cos^2\theta}\right)^2 .$$

When ″$_r$ = 1; = 0; radiation evidently disappears.

The factor A in Eq. (8.194) does not fully agree with that in the original work by Frank and Ginzburg,

$$A' = \frac{1}{1+\beta\cos\theta} + \frac{\Gamma}{1-\beta\cos\theta} - \frac{1+\Gamma}{\varepsilon_r(1+\beta\sqrt{\varepsilon_r - \sin^2\theta})}, \tag{8.195}$$

although this too vanishes when ″$_r$! 1 and reduces to the case of conducting medium when ″$_r$! 1:

8.12 Energy Loss of Charged Particles Moving in Dielectrics

The formula derived in Eq. (8.180) yields a physically meaning energy loss rate even when the Cherenkov condition is not satis ed, < 1: A charged particle moving in a dielectric medium collides with atoms and lose its energy through Coulomb interaction with electrons in atoms. Electrons in an atom are bounded. However, they do respond to electromagnetic disturbance and absorb energy through the resonance ″(!) = 0, where

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2-\omega_0^2}\right): \tag{8.196}$$

Resonance of the type

$$\frac{1}{x-x_0}: \tag{8.197}$$

can be handled mathematically by introducing an imaginary part,

$$\frac{1}{x-x_0} = \mathrm{P}\frac{1}{x-x_0} - i\pi\delta(x-x_0), \tag{8.198}$$

where P stands for the principal part. This is justi able because the imaginary part of the function

$$\frac{1}{x-x_0+i\varepsilon} = \frac{x-x_0}{(x-x_0)^2+\varepsilon^2} - i\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}, \tag{8.199}$$

remains nite even in the limit ″! 0;

$$\lim_{\varepsilon\to 0}\int\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}dx = : \tag{8.200}$$

Physically, the resonance leads to absorption of wave energy by charged particles in a material medium dielectrics, plasmas, etc.

The characteristic scale length of interaction between a charged particle and atoms in a dielectric is evidently of the order of atomic size which indicates that the interaction is of near- eld, nonradiating nature dominated by longitudinal (electrostatic) elds. As we will see, the major contribution to the energy loss occurs through the pole of the dielectric function, ″(!) = 0:

In the near- eld region 1; the modi ed Bessel functions K_0( );K_1( ) may be approximated by

$$K_0(\alpha\rho) \simeq -\ln\left(\frac{\alpha\rho}{2}\right) - \gamma_E, \quad (8.201)$$

$$K_1(\alpha\rho) \simeq \frac{1}{\alpha\rho}, \quad (8.202)$$

where $_E = 0{:}5772$ is the Euler s constant. The real part of Eq. (8.180) becomes

$$\frac{d\mathcal{E}}{dz} \simeq \frac{\mu_0 e^2}{(2\pi)^2} \operatorname{Re} \int_{-\infty}^{\infty} i\omega \left(1 - \frac{1}{\varepsilon(\omega)\mu_0 v^2}\right) \left[\ln\left(\frac{\alpha\rho}{2}\right) + \gamma_E\right] d\omega, \quad (8.203)$$

with

$$= \frac{\omega}{c(\omega)}\sqrt{\frac{1}{\beta^2} - 1}, \ \beta < 1 \, : \quad (8.204)$$

The dielectric function $"(!)$ is in the form

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}\right):$$

Therefore, the integration can be carried out by evaluating the pole contribution at $"(!) = 0;$ which occurs at

$$! = \pm\sqrt{\omega_p^2 + \omega_0^2}, \quad (8.205)$$

and exploiting Plemelj s formula,

$$\lim_{\nu \quad 0} \frac{1}{x - a + i\nu} = \mathrm{P}\frac{1}{x - a} - i\pi\delta(x - a), !$$

where P indicates the principal part of the singular function $1{=}(x \qquad a){:}$ The result is

$$\frac{d\mathcal{E}}{dz} \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{e\omega_p}{v}\right)^2 \ln\left(\frac{1.12v}{\rho\sqrt{\omega_p^2 + \omega_0^2}}\right): \quad (8.206)$$

As the minimum distance ; the intermolecular distance may be substituted because the shell electrons e⁄ectively shield the electric eld of the charge well inside the atom.

In a plasma, $!_0$ is evidently zero (because electrons in a plasma are free). Then,

$$\frac{d\mathcal{E}}{dz} \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{e\omega_p}{v}\right)^2 \ln\left(\frac{v}{\rho_{\min}\omega_p}\right), \quad (8.207)$$

where

$$_{\min} \simeq \left(\frac{3}{4\pi n}\right)^{1/3},$$

is the average distance between ions. For a Maxwellian electron distribution with a temperature $T_e;$ the average energy loss rate may be estimated from

$$-\frac{d\mathcal{E}}{dt} \simeq \frac{1}{4\pi\varepsilon_0}\frac{(e\omega_p)^2}{3v_{Te}}\ln\left(\frac{8\sqrt{2}\pi}{3}n\lambda_D^3\right), \tag{8.208}$$

where

$$\lambda_D = \frac{v_{Te}}{\omega_p}, \tag{8.209}$$

is the Debye shielding length.

8.13 Bremsstrahlung

Whenever a charged particle collides with another charged particle, electromagnetic radiation occurs due to acceleration by Coulomb force. Collisions between like particles (e.g., electron-electron) emit quadrupole radiation while collisions between unlike particles (e.g., electron-ion) emit dipole radiation. Bremsstrahlung is due to collisions between electrons and ions and provides a basic mechanism for x-ray production.

Let an electron approach an ion having a charge Ze with a velocity v (c) and impact parameter b: The acceleration due to Coulomb force is of the order of

$$a \simeq -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{mb^2}; \tag{8.210}$$

and consequent radiation power can be estimated from the Larmor s formula,

$$P \simeq \frac{1}{4\pi\varepsilon_0}\frac{2e^2a^2}{3c^3}: \tag{8.211}$$

The total energy radiated can in principle be found by integrating the power over time along the electron trajectory. However, in experiments, one is seldom interested in measuring radiation power or energy associated with a single electron. What is more relevant is the radiation associated with a beam of electrons impinging on an ion. In this case, some electrons have impact parameters vanishingly small. However, the impact parameter has a lower bound imposed by the uncertainty principle,

$$b_{\min} \simeq \frac{\hbar}{p}, \tag{8.212}$$

where p is the electron momentum. For an impact parameter b; the time duration in which the acceleration is signi cant is

$$\Delta t \simeq \frac{b}{v}: \tag{8.213}$$

Therefore, energy radiated by a single electron is

$$\begin{aligned}\mathcal{E} &= P\Delta t \\ &\simeq \frac{1}{(4\pi\varepsilon_0)^3}\frac{2}{3c^3}\left(\frac{Ze^3}{m}\right)^2\frac{1}{b^3 v}:\end{aligned} \tag{8.214}$$

For an electron beam having a density n; the radiation power can thus be estimated from

$$
\begin{aligned}
P &= n_e v \int_{b_{\min}}^{\infty} \mathcal{E} 2\pi b db \\
&= \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \left(\frac{Ze^3}{m}\right)^2 \frac{n_e m v}{c^3 \hbar} \quad :
\end{aligned}
$$

(8.215)

If the ion density is n_i; the quantity Pn_i de nes the power density,

$$\frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \left(\frac{Ze^3}{m}\right)^2 \frac{n_e n_i m v}{c^3 \hbar}, \quad (\text{W/m}^3): \tag{8.216}$$

The frequency spectrum of radiation energy may qualitatively be found as follows. Since the characteristic time of acceleration

$$= \frac{b}{v}, \tag{8.217}$$

is short, radiation occurs as an impulse and the spectrum is at in the region $0 < ! < v=b;$ and vanishes for $! > v=b:$ Since the minimum impact parameter is

$$b_{\min} \simeq \frac{\hbar}{mv}, \tag{8.218}$$

the upper limit of the frequency spectrum extends to

$$!^{\max} \simeq \frac{mv^2}{\hbar}, \text{ or } \sim ! \,.\, mv^2: \tag{8.219}$$

This is essentially a statement of energy conservation, that is, the maximum photon energy emitted during bremsstrahlung is limited by the incident electron kinetic energy, which is reasonable. Therefore, the frequency spectrum of bremsstrahlung is

$$
I(\omega, b) = \begin{cases} \dfrac{1}{(4\pi\varepsilon_0)^3} \dfrac{2}{3} \dfrac{(Ze^3)^2}{mc^2 m v^2 b^2}, & 0 < \omega < \dfrac{v}{b}, \\[2ex] 0, & \omega > \dfrac{v}{b}. \end{cases} \tag{8.220}
$$

$I(!;b)$ has dimensions of J/frequency. It is convenient to introduce a radiation cross-section $(!)$ de ned by

$$
\begin{aligned}
(\omega) &= \int_{b_{\min}}^{b_{\max}} I(\omega, b) 2\pi b db \\
&= \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2}{mc^2 m v^2} \ln\left(\frac{b_{\max}}{b_{\min}}\right) \\
&= \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2}{mc^2 m v^2} \ln\left(\frac{mv^2}{\hbar\omega}\right) \quad :
\end{aligned} \tag{8.221}
$$

The integral over the frequency,

$$\int (\omega)d\omega; \tag{8.222}$$

evidently diverges at the lower end $\omega \to 0$. To remedy this difficulty, Bethe and Heitler recognized that if the velocity v is understood as the mean value of the initial and final velocities, i.e., before and after emission of a photon,

$$\begin{aligned} v &= \frac{1}{2}(v_{\text{initial}} + v_{\text{final}}) \\ &= \frac{1}{\sqrt{2m}}\left(\sqrt{\mathcal{E}} + \sqrt{\mathcal{E} - \hbar\omega}\right), \end{aligned} \tag{8.223}$$

the integral remains finite,

$$\begin{aligned} \int \chi(\omega)d\omega &= \frac{1}{(4\pi\varepsilon_0)^3}\frac{4\pi}{3}\frac{(Ze^3)^2}{mc^2mv^2}\int_0^{\omega_{\max}} \ln\left(\frac{\left(\sqrt{\mathcal{E}} + \sqrt{\mathcal{E} - \hbar\omega}\right)^2}{\hbar\omega}\right)d\omega \\ &= \frac{1}{(4\pi\varepsilon_0)^3}\frac{4\pi}{3}\frac{(Ze^3)^2}{mc^3\hbar}\int_0^1 \ln\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)dx, \ x = \frac{\hbar\omega}{\mathcal{E}}, \\ &= \frac{1}{(4\pi\varepsilon_0)^3}\frac{4\pi}{3}\frac{(Ze^3)^2}{mc^3\hbar}. \end{aligned} \tag{8.224}$$

The integral in the intermediate step is unity. Multiplying by the ion density n_i; we thus obtain the bremsstrahlung rate per unit length,

$$\begin{aligned} \frac{dE}{dz} &= n_i \int \chi(\omega)d\omega \\ &= \frac{1}{(4\pi\varepsilon_0)^3}\frac{4\pi}{3}\frac{(Ze^3)^2 n_i}{m^2c^3\hbar}, \end{aligned} \tag{8.225}$$

and radiation power density,

$$P/V = \frac{1}{(4\pi\varepsilon_0)^3}\frac{4\pi}{3}\frac{(Ze^3)^2 n_e n_i v}{mc^3\hbar}, \ (\text{W m}^{-3}). \tag{8.226}$$

This agrees with the earlier qualitative estimate in Eq. (8.216).

Relativistic correction to the classical bremsstrahlung formulae can be readily found if we move to the electron frame wherein the electron velocity is nonrelativistic. Since the energy and frequency are Lorentz transformed in the same manner, and the transverse dimensions are Lorentz invariant, it follows that the radiation cross-section (ω) is Lorentz invariant,

$$_{\text{lab}}(\omega_{\text{lab}}) = \ _0(\omega_0);$$

where the primed quantities are those in the electron frame. The frequencies ω_{lab} and ω_0 are related through the relativistic Doppler shift,

$$!_0 = !_{\text{lab}}(1 \qquad \cos); \; !_{\text{lab}} = !_0(1 + \qquad \cos _0); \tag{8.227}$$

where and $_0$ are the angles with respect to the electron velocity in each frame. Since in the electron frame, the radiation is con ned in a small angle about $_0 = =2;$ we have

$$!_0 ' \, \underline{\quad} !_{\text{lab}} : \tag{8.228}$$

The collision time is shortened by the factor since the transverse eld is intensi ed by the same factor through Lorentz transformation. Therefore, the maximum impact parameter is modi ed as

$$b_{\max} = \frac{\gamma v}{\omega'} = \frac{\gamma^2 v}{\omega_{\text{lab}}}, \tag{8.229}$$

and the radiation cross-section in the laboratory frame becomes

$$(\omega) = \chi'(\omega') = \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{e^4 (Ze)^2}{m^2 c^3 v^2} \ln\left(\frac{\gamma^2 m v^2}{\hbar\omega}\right) : \tag{8.230}$$

It is noted that the minimum impact parameter remains unchanged through the transformation because it is essentially the Compton length based on the uncertainty principle.

8.14 Radiation due to Electron-Electron Collision

In this case the dipole radiation is absent because in the center of mass frame, two electrons stay at opposite positions, $r_1 = r_2;$ and the dipole moment identically vanishes.. The lowest order radiation process is that due to electric quadrupole. (The magnetic dipole moment also vanishes.) The quadrupole moment tensor is

$$Q_{ij} = \frac{1}{2} e x_i x_j, \tag{8.231}$$

where x_i is the i-th component of the relative distance $2\mathbf{r}:$ In Chapter 5, a general formula for the quadrupole radiation power has been derived. Noting

$$\frac{d^3}{dt^3}(x_i x_j) = \dddot{x}_i x_j + 3\ddot{x}_i \dot{x}_j + 3\dot{x}_i \ddot{x}_j + x_i \dddot{x}_j, \tag{8.232}$$

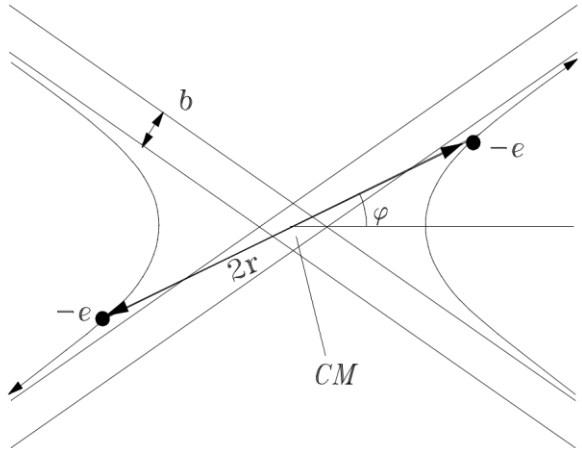


Figure 8-11: Colliding electrons in the center of mass frame. The impact parameter is b:

$$\ddot{x}_i = \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{mr^3}x_i, \quad (8.233)$$

$$\dddot{x}_i = \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{m}\left(\frac{x_i}{r^3} - \frac{3x_i v_r}{r^4}\right), \quad (8.234)$$

we nd

$$\frac{d^3}{dt^3}(x_i x_j) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{mr^3}\left(4(v_i x_j + v_j x_i) - \frac{6v_r x_i x_j}{r}\right), \quad (8.235)$$

$$\dddot{Q} = \frac{1}{4\pi\varepsilon_0}\frac{e^3}{mr^3}\left(4(v_i x_j + v_j x_i) - \frac{6v_r x_i x_j}{r}\right) \quad (8.236)$$

Substituting this into the quadrupole radiation power,

$$P = \frac{1}{4\pi\varepsilon_0}\frac{1}{60c^5}\left[3\sum_{ij}\dddot{Q}_{ij}^2 - \left(\sum_i \dddot{Q}_{ii}\right)^2\right], \quad (8.237)$$

we obtain, after somewhat lengthy calculations,

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{15c^5}\left(\frac{e^3}{m}\right)^2\frac{v^2 + 11v_\phi^2}{r^4}, \quad (8.238)$$

where v is the component of the velocity related to the initial angular momentum $bv_0 = r(t)v\ (t)$ with v_0 the velocity at r ! 1: Energy conservation reads

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r}; \quad (8.239)$$

Then

$$v^2 + 11v_\phi^2 = v_0^2 - \frac{4e^2}{4\pi\varepsilon_0 mr} + 11\frac{(bv_0)^2}{r^2}, \quad (8.240)$$

and the total radiation energy can be found by integrating the radiation power over time along the trajectory,

$$\mathcal{E} = \frac{1}{4\pi\varepsilon_0}\frac{2}{15c^5}\left(\frac{e^3}{m}\right)^2\int_{-\infty}^{\infty}\frac{1}{r^4}\left(v_0^2 - \frac{4e^2}{4\pi\varepsilon_0 mr} + 11\frac{(bv_0)^2}{r^2}\right)dt. \quad (8.241)$$

This can be converted to an integral over the distance r by noting

$$dt = \frac{dr}{v_r} = \frac{dr}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi\varepsilon_0 mr}}}, \quad (8.242)$$

$$\mathcal{E} = \frac{1}{4\pi\varepsilon_0}\frac{2}{15c^5}\left(\frac{e^3}{m}\right)^2 \times 2\int_{r_{\min}}^{\infty}\frac{1}{r^4}\frac{v_0^2 - \frac{4e^2}{4\pi\varepsilon_0 mr} + 11\frac{(bv_0)^2}{r^2}}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi\varepsilon_0 mr}}}dr, \quad \text{(J)}$$

(8.243)

where $r_{\min}$ is the distance of the closest approach,

$$r_{\min} = \frac{1}{2v_0^2}\left(\frac{4e^2}{m} + \sqrt{\left(\frac{4e^2}{m}\right)^2 + 4(bv_0)^2}\right): \quad (8.244)$$

However, the radiation energy by a single electron pair is of no practical interest. What is more relevant is the radiation power emitted by an electron beam impinging on a single electron which can be evaluated from

$$\begin{aligned} P &= 2\pi n v_0\int_0^{\infty}\mathcal{E}(b)bdb \\ &= \frac{1}{4\pi\varepsilon_0}\frac{4}{15c^5}\left(\frac{e^3}{m}\right)^2 2\pi n v_0\int_0^{\infty}bdb\int_{r_{\min}}^{\infty}\frac{1}{r^4}\frac{v_0^2 - \frac{4e^2}{4\pi\varepsilon_0 mr} + 11\frac{(bv_0)^2}{r^2}}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi\varepsilon_0 mr}}}dr, \text{ (W)}. \quad (8.245) \end{aligned}$$

The double integral reduces to

$$\begin{aligned} &\int_0^{\infty}bdb\int_{r_{\min}}^{\infty}dr\frac{1}{r^4}\frac{v_0^2 - \frac{4e^2}{4\pi\varepsilon_0 mr} + 11\frac{(bv_0)^2}{r^2}}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi\varepsilon_0 mr}}} \\ &= \frac{25}{3}v_0\int_{r_{\min}}^{\infty}\frac{1}{r^5}\left(r^2 - \frac{4re^2}{mv_0^2}\right)^{3/2}dr = \frac{5}{6}\frac{mv_0^3}{e^2}. \end{aligned} \quad (8.246)$$

Therefore,

$$P = \frac{1}{4\pi\varepsilon_0}\frac{4\pi}{9}\frac{ne^4v_0^4}{mc^5}, \text{(W)}. \quad (8.247)$$

This de nes a radiation cross-section,

$$= \frac{P}{\frac{1}{2}nmv_0^3} = \frac{8\pi}{9}\frac{v}{c}r_e^2, \ (\mathrm{m}^2) \tag{8.248}$$

where $\frac{1}{2}nmv_0^3$ is the energy ux density of the beam and

$$r_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2} = 2.85 \times 10^{-15} \ \mathrm{m},$$

is the classical radius of electron.