Small Oscillations

Here we consider small oscillations of mechanical systems about their equilibrium states. We will see that as long as the amplitude of the oscillations is small enough, the motion demonstrates an amazingly simple and generic character. The system behaves like a set of independent one-dimensional oscillators. The generalized coordinates describing each oscillator are called normal modes. The machinery being developed below is all about identifying the normal modes and their parameters— frequencies of their oscillations and their relationship to the original coordinates.

Bilinearized Lagrangian

In this chapter we confine ourselves with the Lagrangians that microscopically originate from a system of particles interacting through central potentials:

$$L = T - U\,, \qquad T = \sum_{i=1}^{N} \frac{m_i \dot{\mathbf{r}}_i^2}{2}\,, \qquad U = \sum_{i<j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)\,. \tag{1}$$

First, we make certain observations concerning the form of the Lagrangian (1) in new generalized coordinates, $\{q_\alpha\}$, related to the Cartesian coordinates by some functions:

$$q_\alpha = f_\alpha(\{\mathbf{r}_i\})\,. \tag{2}$$

From (2) we see that

$$\dot{q}_\alpha = \sum_{i=1}^{N} \frac{\partial f_\alpha}{\partial \mathbf{r}_i} \cdot \dot{\mathbf{r}}_i\,, \tag{3}$$

and this implies that $\dot{\mathbf{r}}_i$ is a *linear* function of $\dot{q}$'s with some coefficients which are the functions of q's. Correspondingly, the form of the Lagrangian in new coordinates is as follows.

$$L = \frac{1}{2}\sum_{\alpha\beta} m_{\alpha\beta}\,\dot{q}_\alpha \dot{q}_\beta - U(\{q\})\,, \tag{4}$$

where $m_{\alpha\beta}$ is a matrix which depends on q's, but not on $\dot{q}$'s. By definition, the equilibrium is the time-independent solution of the equations of motion (for all α's):

$$q_\alpha(t) = q_\alpha^{(0)}, \qquad \dot{q}_\alpha \equiv 0\,. \tag{5}$$

In accordance with the Lagrange equations, the equilibrium values $q_\alpha^{(0)}$ are found from the system of equations (for all α's)

$$\frac{\partial U(\{q^{(0)}\})}{\partial q_\alpha^{(0)}} = 0\,. \tag{6}$$

Introducing the deviations of the coordinates from their equilibrium positions,

$$x_\alpha(t) = q_\alpha(t) - q_\alpha^{(0)}, \qquad \dot{x}_\alpha = \dot{q}_\alpha, \tag{7}$$

and taking into account that we are interested only in the limit of small x's, we expand the Lagrangian as the Taylor series in powers of x's up to the leading terms. In the kinetic-energy term, this amounts to just requiring that

$$m_{\alpha\beta}(\{q\}) = m_{\alpha\beta}(\{q^{(0)}\}) . \tag{8}$$

In the potential-energy term we have to expand up to the second-order partial derivatives, since the first-order derivatives are equal to zero by the condition of equilibrium, Eq. (6), and the first term of the Taylor series is a dynamically irrelevant constant. Omitting the latter constant, we arrive at the *bilinearized* Lagrangian

$$L = (1/2) \sum_{\alpha,\beta} [m_{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta - k_{\alpha\beta} x_\alpha x_\beta] , \tag{9}$$

in which $m_{\alpha\beta}$ and $k_{\alpha\beta}$ are certain constant matrices, and, in particular

$$k_{\alpha\beta} = \frac{\partial^2 U(\{q^{(0)}\})}{\partial q_\alpha^{(0)} \partial q_\beta^{(0)}} . \tag{10}$$

Without loss of generality, we will be assuming that matrices $m_{\alpha\beta}$ and $k_{\alpha\beta}$ are symmetric, since one can always symmetrize them without changing L. [The symmetrization is done by switching dummy subscripts α and β.]

The structure of Eq. (9) suggests utilizing vector-matrix notation. Introducing the vector

$$\sim x = (x_1, x_2, x_3, ...) , \tag{11}$$

and the matrices M and K, such that

$$(\mathrm{M})_{\alpha\beta} = m_{\alpha\beta}, \qquad (\mathrm{K})_{\alpha\beta} = k_{\alpha\beta}, \tag{12}$$

we write

$$L = \frac{1}{2} \left[\dot{\vec{x}} \cdot (\mathcal{M} \dot{\vec{x}}) - \vec{x} \cdot (\mathcal{K} \vec{x}) \right] . \tag{13}$$

Solving the Equation of Motion

The Lagrange equations of motion read

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\partial L}{\partial \vec{x}} . \tag{14}$$

From (13) we find

$$\frac{\partial L}{\partial \dot{\vec{x}}} = \mathrm{M} \sim \dot{x} , \qquad \frac{\partial L}{\partial \vec{x}} = -\mathcal{K} \vec{x} , \tag{15}$$

and get the equation of motion in the form

$$\mathrm{M} \sim \ddot{x} + \mathrm{K} \sim x = 0 . \tag{16}$$

Mathematically, we are dealing with a second-order linear differential equation with constant coefficients, the general solution to which comes in the form of a linear combination—with arbitrary coefficients playing the role of free constants of integration—of special solutions:

$$\vec{x}(t) \;=\; \sum_s C_s\, \vec{a}_s(t) \qquad (17)$$

Here C_s are arbitrary coefficients and

$$\sim a_s(t) = \sim u_s\, \mathrm{Re}\, e^{i(\omega_s t - \delta_s)} = \sim u_s \cos(\omega_s t - \delta_s) \qquad (18)$$

is the special solution, where $\sim u_s$ is a real eigenvector, ω_s is an eigenfrequency, and δ_s is an arbitrary phase shift. In view of the special form of time-dependent factor in Eq. (18), the linear differential equation for $\sim a_s(t)$,

$$\mathrm{M}\sim \ddot{a}_s + \mathrm{K}\sim a_s = 0\,, \qquad (19)$$

reduces to linear algebraic equation for $\sim u_s$:

$$(\mathcal{K} \;-\; \omega_s^2\, \mathcal{M})\, \vec{u}_s \;=\; 0\,, \qquad (20)$$

which has non-trivial solutions only for special values of the parameter ω_s, when the determinant of the matrix is equal to zero

$$\det\,(\mathcal{K} \;-\; \omega_s^2\, \mathcal{M}) \;=\; 0\,. \qquad (21)$$

The latter requirement forms the equation for finding eigenfrequencies ω_s, the so-called *characteristic equation*. For each eigenfrequency ω_s, Eq. (20) then gives one or more linear independent eigenvectors. (The total number of linear independent eigenvectors is always equal to the dimensionality of the original vector $\sim x$.)

Normal Modes

With Eq. (18) taken into account, the solution (17) can be written as

$$\vec{x}(t) \;=\; \sum_s \vec{u}_s\, \eta_s(t)\,, \qquad (22)$$

where

$$\eta_s(t) = C_s\, \mathrm{Re}\, e^{i(\omega_s t - \delta_s)} = C_s \cos(\omega_s t - \delta_s)\,. \qquad (23)$$

Now we realize that actually we could do a better job if from the very beginning instead of solving the equation of motion we simply introduced new generalized coordinates, $\{\eta_s\}$, related to the original ones by

$$\vec{x} \;=\; \sum_s \vec{u}_s\, \eta_s \qquad (24)$$

Indeed, Eq. (23) shows that these coordinates are *independent* from each other, and are nothing else than coordinates of harmonic oscillators. These coordinates are called *normal modes.*

Let us see how one arrives at the normal modes directly from the bilinearized Lagrangian. The starting point is the transformation (24) with the requirement that the linear independent constant vectors $\sim u_s$ are obtained from solving the problem (20)-(21). Note that the eigenfrequencies ω_s arise as simply the eigenvalues of this problem.—At this point we do not ascribe to them any physical meaning. When all linear independent $\sim u_s$'s are found, we plug (24) into the bilinearized Lagrangian (13) to get

$$L = \frac{1}{2}\sum_{s,s'}\left[\dot{\eta}_s\dot{\eta}_{s'}\,\vec{u}_s\cdot(\mathcal{M}\vec{u}_{s'}) - \eta_s\eta_{s'}\,\vec{u}_s\cdot(\mathcal{K}\vec{u}_{s'})\right] . \tag{25}$$

Now we take into account two crucial mathematical properties of the solutions of the problem (20)(21). Namely,

$$\sim u_s\cdot(\mathrm{M}\sim u_{s0}) = 0\,, \qquad s \neq s^0\,, \tag{26}$$

$$\sim u_s\cdot(\mathrm{K}\sim u_{s0}) = 0\,, \qquad s \neq s^0\,. \tag{27}$$

We will prove these relations a bit later, and now just use them in Eq. (25)

$$L = \sum_s \frac{m_s}{2}\left[\dot{\eta}_s^2 - \omega_s^2\,\eta_s^2\right], \tag{28}$$

with

$$m_s = \sim u_s\cdot(\mathrm{M}\sim u_s)\,. \tag{29}$$

We have also took into account that in accordance with (20)

$$\sim u_s\cdot(\mathrm{K}\sim u_s) = \omega_s^2\sim u_s\cdot(\mathrm{M}\sim u_s)\,. \tag{30}$$

Actually, since the vectors $\sim u_s$ are defined by Eq. (20) only up to a normalization, it makes sense to normalize them in such a way that

$$\sim u_s\cdot(\mathrm{M}\sim u_s) = 1\,, \tag{31}$$

in which case the Lagrangian acquires most simple form

$$L = \frac{1}{2}\sum_s\left[\dot{\eta}_s^2 - \omega_s^2\,\eta_s^2\right]. \tag{32}$$

The global coefficient 1/2 can be actually omitted since it does not change the equations of motion.

Now we prove relations (26)-(27). From (20) we have

$$\mathcal{K}\,\vec{u}_s = \omega_s^2\,\mathcal{M}\,\vec{u}_s, \tag{33}$$

$$\mathcal{K}\,\vec{u}_{s'} = \omega_{s'}^2\,\mathcal{M}\,\vec{u}_{s'}. \tag{34}$$

Forming corresponding inner products, we get

$$\sim u_{s0}\cdot(\mathrm{K}\sim u_s) = \omega_s^2\sim u_{s0}\cdot(\mathrm{M}\sim u_s)\,, \tag{35}$$

$$\sim u_s\cdot(\mathrm{K}\sim u_{s0}) = \omega_{s^0}^2\sim u_s\cdot(\mathrm{M}\sim u_{s0})\,. \tag{36}$$

By the symmetry of the matrices K and M, for any two vectors $\sim a$ and $\sim b$

$$\sim a\cdot(\mathrm{K}\sim b) = \sim b\cdot(\mathrm{K}\sim a)\,, \qquad \sim a\cdot(\mathrm{M}\sim b) = \sim b\cdot(\mathrm{M}\sim a)\,. \tag{37}$$

This allows us to combine (35)-(36) into

$$(\omega_s^2 - \omega_{s^0}^2)\sim u_{s0}\cdot(\mathrm{M}\sim u_s) = 0\,, \tag{38}$$

$$(\omega_s^2 - \omega_{s^0}^2)\sim u_{s0}\cdot(\mathrm{K}\sim u_s) = 0\,. \tag{39}$$

If $\omega_{s'}^2 \neq \omega_s^2$, this immediately proves the theorem. The case when $\omega_{s'}^2 = \omega_s^2$, the so-called degenerate eigenvalue, is a bit more subtle. In this case there exist more than one linear independent solutions with one and the same eigenvalue. However, one can always *select* solutions in such a form that Eqs. (26)-

(27) are satisfied. Indeed, the degeneracy of the eigenfrequencies can be lifted by just an infinitesimally small correction to either the matrix M or the matrix K, and the solutions to the new problem will automatically satisfy Eqs. (26)-(27), while being only infinitesimally different from *some* solutions of the original problem.

Example: Three-atomic one-dimensional "Molecule".

Consider a one-dimensional chain of three atoms with masses m_1, m_2, m_3 bounded by just two potentials: The potential $U_{12}(|x_1 - x_2|)$ binds the first and the second atoms while the potential $U_{23}(|x_2 - x_3|)$ binds the second and the third atom. For simplicity, there is no direct interaction between the first and the third atoms. What are the normal modes of the small oscillations of this system?

We start with the Lagrangian in Cartesian coordinates.

$$L = \sum_{i=1}^{3} \frac{m_i}{2} \dot{x}_i^2 - U_{12}(|x_1 - x_2|) - U_{23}(|x_2 - x_3|) . \tag{40}$$

We need to bilinearize this Lagrangian in terms of the deviations of the coordinates from their equilibrium positions:

$$\xi_i = x_i - x^{(0)}{}_i \qquad (i = 1,2,3) , \tag{41}$$

where $x^{(0)}{}_i$'s are the equilibrium positions given by the equations (6) that in our case reduce to

$$U'_{12}(|x_1^{(0)} - x_2^{(0)}|) = 0 , \qquad U'_{23}(|x_2^{(0)} - x_3^{(0)}|) = 0 . \tag{42}$$

The kinetic part of the Lagrangian is already bilinear, and we just replace $\dot{x}_i \to \dot{\xi}_i$.

$$T = \sum_{i=1}^{3} \frac{m_i}{2} \dot{\xi}_i^2 . \tag{43}$$

For the matrix M we thus have

$$\mathrm{M}_{ii} = m_i , \qquad \mathrm{M}_{ij} = 0 , \qquad \text{if } i \neq j . \tag{44}$$

In the potential-energy part of the Lagrangian, we expand pair potentials as Taylor series up to the terms with second derivatives. We get

$$U_{12}(|x_1 - x_2|) \approx \text{const} + \frac{\kappa_{12}}{2} (\xi_1 - \xi_2)^2 , \qquad \kappa_{12} = U''_{12}(|x_1^{(0)} - x_2^{(0)}|) , \tag{45}$$

$$U_{23}(|x_2 - x_3|) \approx \text{const} + \frac{\kappa_{23}}{2} (\xi_2 - \xi_3)^2 , \qquad \kappa_{23} = U''_{23}(|x_2^{(0)} - x_3^{(0)}|) . \tag{46}$$

Omitting the constants, we write the total potential energy as

$$U = (1/2)[\kappa_{12}\xi_{12} + (\kappa_{12} + \kappa_{23})\xi_{22} + \kappa_{23}\xi_{32} - 2\kappa_{12}\xi_1\xi_2 - 2\kappa_{23}\xi_2\xi_3] , \tag{47}$$

which yields the following structure of the matrix K (remember that the symmetrization is a must!):

$$\mathrm{K}_{11} = \kappa_{12} , \qquad \mathrm{K}_{12} = -\kappa_{12} , \qquad \mathrm{K}_{13} = 0 , \tag{48}$$

$$K_{21} = -\kappa_{12}, \qquad K_{22} = \kappa_{12} + \kappa_{23}, \qquad K_{23} = -\kappa_{23}, \tag{49}$$

$$K_{31} = 0, \qquad K_{32} = -\kappa_{23}, \qquad K_{33} = \kappa_{23}. \tag{50}$$

Now we are in a position to construct the characteristic equation (we use $\lambda = \omega^2$)

$$\det(K - \lambda M) = 0. \tag{51}$$

While doing the determinant it is convenient to absorb λ into M, and then restore it in the final answer by grouping the terms with the same power of the masses. We arrive at the following characteristic equation

$$(A\lambda^2 + B\lambda + C)\lambda = 0, \tag{52}$$

where

$$A = m_1 m_2 m_3, \tag{53}$$

$$B = -[\kappa_{12} m_3 (m_1 + m_2) + \kappa_{23} m_1 (m_2 + m_3)], \tag{54}$$

$$C = \kappa_{12} \kappa_{23} (m_1 + m_2 + m_3). \tag{55}$$

Note that one of the solutions is trivial: $\lambda = 0$. This "pathological" solution is a consequence of translational and Galilean symmetry of the problem. One of the degrees of freedom is just the constant-velocity motion of the center of mass. It can be viewed as an oscillator with zero frequency. The other two normal modes are non-trivial. Their frequencies are given by the equation

$$A\lambda^2 + B\lambda + C = 0. \tag{56}$$

Corresponding two eigenvectors are then found from Eq. (20).

Problem 33. The systems described by Lagrangian

$$L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 - \frac{\kappa_{11}}{2}x_1^2 - \frac{\kappa_{22}}{2}x_2^2 - \frac{\kappa_{12}}{2}(x_1 - x_2)^2. \tag{57}$$

are generically referred to as *two coupled harmonic oscillators*.

(a) Give a particular example of a mechanical system described by this Lagrangian.
(b) Find the eigenfrequencies and eigenvectors of the normal modes (the particular normalization of the eigenvectors is up to you).
(c) Interpret the structure of the normal modes in the case $m_1 \text{ ¿ } m_2$ with $\kappa_{11} = \kappa_{22} = \kappa_{12} = \kappa$.

Problem 34. The system of two coupled plane pendulums described in Problem 26 performs small oscillations about its equilibrium position $\theta_1 = \theta_2 = 0$.

(a) Find the eigenfrequencies and eigenvectors (the particular normalization of the eigenvectors is up to you) of the normal modes.
(b) Interpret the structure of the normal modes in the case $m_1 \text{ ¿ } m_2$ with $l_1 = l_2 = l$.
(c) Interpret the structure of the normal modes in the case $m_1 \text{ À } m_2$ with $l_1 = l_2 = l$.

Driven Motion of Normal Modes

Normal modes can be excited by applying a small external perturbation. In terms of Lagrangian, such a perturbation corresponds to a time-dependent generalized external-potential term:

$$L \to L - U_{\text{ext}}(t,\{q\})\,. \tag{58}$$

Since the normal oscillations imply the limit of small amplitudes, the potential U_{ext} should be Taylor expanded up to the first not-trivial terms about the position of equilibrium $\{q^{(0)}\}$:

$$U_{\text{ext}}(t,\{q\}) \to U_{\text{ext}}(t,\{q^{(0)}\}) - f\tilde{}(t)\cdot\tilde{}x\,, \tag{59}$$

where

$$f_\alpha(t) = -\frac{\partial U_{\text{ext}}(t,\{q\})}{\partial q_\alpha} \qquad (\text{at } \{q\} = \{q^{(0)}\}) \tag{60}$$

can be interpreted as the generalized force acting on the variable x_α. Note that the first term in the r.h.s. of Eq. (59) is trivial. It is just a function of time that does not effect the equations of motion. We omit it from now on.

Using relation (24) between the original and normal variables, we obtain the Lagrangian for the normal modes

$$L = \sum_s \left[\dot{\eta}_s^2/2 - \omega_s^2\,\eta_s^2/2 + f_s(t)\,\eta_s\right], \tag{61}$$

where

$$f_s(t) = f\tilde{}(t)\cdot\tilde{}u_s \tag{62}$$

has the meaning of the force acting on the normal variable η_s. The structure of the Lagrangian means that each normal mode moves independently from the others being driven by its own force. The equation of motion for the s-th normal mode is readily obtained. Below we omit the subscript s for the mode (and also introduce the subscript 0 to distinguish the eigenfrequency of the mode from the frequency of the periodic driving force).

$$\ddot{\eta} + \omega_0^2\,\eta = f(t)\,. \tag{63}$$

The solution to this linear differential equation can be written as

$$\eta = \eta_1 + \eta_2\,, \tag{64}$$

where η_1 is the general (with free constants of integration) solution to the homogeneous equation

$$\ddot{\eta}_1 + \omega_0^2\,\eta_1 = 0\,, \tag{65}$$

while η_2 is a particular (no free constants) solution to the original equation. Since Eq. (65) is the one-dimensional harmonic oscillator which we have already discussed, we just write down the answer in three equivalent forms

$$\eta_1(t) = A\cos(\omega_0 t - \phi) \equiv A\sin(\omega_0 t - \phi\tilde{}) \equiv A_1\cos\omega_0 t + A_2\sin\omega_0 t\,. \tag{66}$$

Let us find the solution η_2 for the most important case of a sinusoidal force

$$f(t) = a\cos\omega t \equiv \text{Re}\,a e^{i\omega t}. \tag{67}$$

It is convenient to look for the solution in the exponential form

$$\eta_2 = \text{Re}\,B e^{i\omega t}. \tag{68}$$

This substitution translates the original differential equation into an algebraic equation for B yielding

$$B = \frac{a}{\omega_0^2 - \omega^2}, \tag{69}$$

and thus

$$\eta_2 = \frac{a}{\omega_0^2 - \omega^2} \cos \omega t \, . \tag{70}$$

At $\omega \to \omega_0$ the amplitude of the solution becomes arbitrarily large. This is the phenomenon of resonance. Let us trace the initial period of evolution from some small amplitude at $t = 0$ [for simplicity, we just set $\eta(t = 0) = 0$] to a large amplitude of Eq. (70). The relevant solution is

$$\eta(t) = \frac{a}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t] \, . \tag{71}$$

At small enough times when

$$t \text{ ¿ } 1/|\omega_0 - \omega| \tag{72}$$

we have $\cos\omega t \approx \cos\omega_0 t$ and can take advantage of Taylor expansion

$$\cos\omega t = \cos\omega_0 t + (\omega_0 - \omega) t \sin\omega_0 t + \ldots \tag{73}$$

to get

$$\eta(t) \approx \frac{at}{2\omega_0} \sin \omega_0 t \, , \qquad (\, t \text{ ¿ } 1/|\omega_0 - \omega| \,) \, . \tag{74}$$

We see that the amplitude grows linearly up to times $t \sim 1/|\omega_0 - \omega|$ when the character of motion should change. (At $\omega = \omega_0$, equation (74) is the exact solution for any t.) To analyze the close-toresonance solution at larger times, we write Eq. (71) in complex form

$$\eta(t) = \frac{a}{\omega_0^2 - \omega^2} \mathrm{Re}[e^{i\omega t} - e^{i\omega 0 t}] \tag{75}$$

and note that

$$e^{i\omega t} - e^{i\omega_0 t} = e^{i(\omega+\omega_0)t/2} \left[e^{i(\omega-\omega_0)t/2} - e^{i(\omega_0-\omega)t/2} \right] = 2ie^{i(\omega+\omega_0)t/2} \sin \frac{\omega - \omega_0}{2} t \, . \tag{76}$$

Taking the real part then readily leads to the answer

$$\eta(t) = \frac{2a}{\omega_0^2 - \omega^2} \sin \frac{\omega_0 - \omega}{2} t \, \sin \frac{\omega + \omega_0}{2} t \, . \tag{77}$$

We are interested in the case $|\omega_0 - \omega| \text{ ¿ } \omega_0$. In this case the first sine can be viewed as a slowly varying time-dependent complex amplitude $A(t)$ which absolute value $|A|$ slowly oscillates between zero and 1, modulating the sinusoidal oscillation of the normal mode. [The latter oscillates with the frequency $(\omega + \omega_0)/2 \approx \omega_0$.] This phenomenon is known as *beats*.

Damped Harmonic Oscillator

Dissipative processes cannot be directly taken into account by Lagrangian formalism. The problem of microscopic description of the phenomenon of dissipation is quite non-trivial and goes far beyond this course. Nevertheless, at the phenomenological level, the dissipation (friction) can be readily taken into

account in the equations of motion. For the harmonic oscillator this is done by adding a force linearly proportional (and oppositely directed) to the velocity:

$$\ddot{\eta} + \omega_0^2 \eta + \gamma \dot{\eta} = f(t) . \tag{78}$$

The proportionality coefficient γ is called friction coefficient. Its dimensionality is the same as the dimensionality of ω_0, but physical meaning is quite different. While the term with ω_0 is responsible for the oscillations, the term with γ leads to *damping* of the oscillatory motion.

Mathematically, the procedure of solving Eq. (78) is identical to the one considered above for the $\gamma = 0$ case. The solution can be written as the superposition, $\eta = \eta_1 + \eta_2$, of the general solution of the homogeneous equation

$$\ddot{\eta}_1 + \omega_0^2 \eta_1 + \gamma \dot{\eta}_1 = 0 \tag{79}$$

and a particular solution, η_2, of the original equation. The physics of the dissipative problem, however, is such that the general solution of Eq. (79) vanishes with time, so that after a certain transient regime only the particular solution survives. That is why finding the particular solution is of prime importance, and we begin our analysis with it. Assuming that the force is given by Eq. (67), we look for the solution in the form (68). Substitution into the equation of motion yields

$$B = \frac{a}{\omega_0^2 - \omega^2 + i\gamma\omega} \qquad \Rightarrow \qquad |B| = \frac{a}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} . \tag{80}$$

If $\gamma \lesssim \omega_0$, then the amplitude $|B|$ has a peaked shape with the characteristic width of the order of γ. The resonance takes place at the frequency $\omega_R \approx \omega_0$ and the amplitude at resonance is perfectly approximated by the value of $|B|$ at $\omega = \omega_0$:

$$|B|_R \approx \frac{a}{\omega_0 \gamma} \qquad (\gamma \lesssim \omega_0) . \tag{81}$$

With increasing γ the peak becomes broader and its amplitude decreases. At $\gamma \sim \omega_0$ the position of the peak significantly shifts from ω_0 to lower frequencies, as is seen from the exact formula for the resonance frequency (the frequency corresponding to the maximal $|B|$):

$$\omega_R = \sqrt{\omega_0^2 - \gamma^2/2} . \tag{82}$$

From this relation we also see that the position of the peak reaches zero frequency at $\gamma = \sqrt{2}\,\omega_0$.

Problem 35. Derive Eq. (82) and find the exact relation for $|B|_R$. Show that at $\gamma > 2\omega_0$ the maximal $|B|$ corresponds to $\omega = 0$, and $|B|$ monotonically decreases with increasing ω.

Turning from the complex amplitude (80) to the physical real-valued solution one finds

$$\eta_2(t) = \frac{a}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \cos(\omega t - \varphi_0) , \tag{83}$$

where the phase shift ϕ_0 is given by

$$\tan \varphi_0 = \frac{\gamma\omega}{\omega_0^2 - \omega^2} . \tag{84}$$

Problem 36. Derive Eq. (84).

Finally, we present the answer for the general solution $\eta_1(t)$ which is important for describing the transient behavior of the forced system (especially pronounced at $\gamma \lesssim \omega_0$). This solution also describes the relaxation of the damped oscillator in the absence of the external drive.

There are two physically different cases: (i) the *underdamped* case of $\gamma < 2\omega_0$ and (ii) the *overdamped* case of $\gamma > 2\omega_0$. In the underdamped case the system keeps oscillating with exponentially decaying amplitude of oscillations:

$$\eta_1(t) = Ae^{-\gamma t/2}\cos(\tilde{\omega}t - \phi_0)\,, \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2/4} \qquad (\gamma < 2\omega_0)\,. \tag{85}$$

Note a non-trivial fact that the frequency of oscillations $\tilde{\omega}$ depends on γ and *vanishes* as $\gamma \to 2\omega_0$.

In the overdamped case there are no oscillations. The solution now becomes

$$\eta_1(t) = A_1\,e^{-\lambda_1 t} + A_2\,e^{-\lambda_2 t}\,, \qquad \lambda_{1,2} = \frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2} \qquad (\gamma > 2\omega_0)\,. \tag{86}$$

Problem 37. Derive Eqs. (85)-(86) and also consider the special case $\gamma = 2\omega_0$ by carefully doing the limits $\gamma \to 2\omega_0 \pm 0$ (Taylor expanding the solutions). Note that it is not good enough to simply set $\gamma = 2\omega_0$ in the solution (85) or (86) since this way you will miss one of the two linear independent parts of the solution, or, equivalently, one of the two free constants of integration. Check that the solution you find at $\gamma = 2\omega_0$ does have two independent free constants and does satisfy the equation (79).