Vector and Scalar Potentials

Maxwell Equations

The electromagnetic field is described by two vector fields: the electric field intensity $\mathcal{E}$ and the magnetic field intensity $\boldsymbol{H}$, which both depend on their position in space (Cartesian coordinates x, y, and z) and time t. The vectors $\mathcal{E}$ and $\boldsymbol{H}$ are determined by the electric charges and their currents. The charges are defined by the charge density function $\rho(x, y, z,t)$, such that $\rho(x, y, z,t)dV$ at time t represents the charge in the infinitesimal volume dV that contains the point (x, y, z). The velocity of the charge in position x, y, z measured at time t represents the vector field $\boldsymbol{v}(x, y, z,t)$, while the current in point x, y, z measured at t is equal to $\boldsymbol{i}(x, y, z,t) = \rho(x, y, z,t)\boldsymbol{v}(x, y, z,t)$.

It turned out (as shown by James Maxwell), that $\boldsymbol{H}$, $\mathcal{E}$, ρ, and $\boldsymbol{i}$ are interrelated by the *Maxwell equations* (c stands for the speed of light):

$$\nabla \times \mathcal{E} + \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = \boldsymbol{0}, \tag{G.1}$$

$$\nabla \times \boldsymbol{H} - \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{i}, \tag{G.2}$$

$$\nabla \cdot \mathcal{E} = 4\pi\rho, \text{ and} \tag{G.3}$$

$$\nabla \cdot \boldsymbol{H} = 0. \tag{G.4}$$

The Maxwell equations have an alternative notation, which involves two new quantities: the *scalar potential* ϕ and the *vector potential* $\boldsymbol{A}$, which replace $\mathcal{E}$ and $\boldsymbol{H}$:

$$\mathcal{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \tag{G.5}$$

$$\boldsymbol{H} = \nabla\times\boldsymbol{A}. \tag{G.6}$$

After inserting E and $\boldsymbol{H}$ from the last equation into the Maxwell equations (G.1) and (G.4), we obtain their automatic satisfaction (for smooth vector components):

$$\nabla\times\mathrm{E} + \frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} = \nabla\times\left(-\nabla\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right) + \frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} = -\nabla\times\nabla\phi - \frac{1}{c}\frac{\partial\nabla\times\boldsymbol{A}}{\partial t} + \frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} = \mathbf{0}$$

and

$$\nabla\cdot(\nabla\times\boldsymbol{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0,$$

because[1] $\nabla\times\nabla\phi = 0$ and $\nabla\times\boldsymbol{A} = \boldsymbol{H}$, while Eqs. (G.2) and (G.3) transform into

$$\nabla\times\nabla\times\boldsymbol{A} + \frac{1}{c}\frac{\partial\nabla\phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \frac{4\pi}{c}\boldsymbol{i}$$

$$-\nabla\cdot(\nabla\phi) - \frac{1}{c}\frac{\partial\nabla\cdot\boldsymbol{A}}{\partial t} = 4\pi\rho,$$

which, in view of the identity $\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \Delta\boldsymbol{A}$ and $\nabla\cdot(\nabla\phi) = \Delta\phi$, gives two additional Maxwell equations [besides Eqs. (G.5) and (G.6)]:

$$\nabla\left(\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) - \Delta\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \frac{4\pi}{c}\boldsymbol{i} \tag{G.7}$$

$$\Delta\phi + \frac{1}{c}\nabla\cdot\frac{\partial\boldsymbol{A}}{\partial t} = -4\pi\rho. \tag{G.8}$$

To characterize the electromagnetic field, we may use either E and $\boldsymbol{H}$ or the two potentials: ϕ and $\boldsymbol{A}$.

[1] $\nabla\times\nabla f = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right] = [0, 0, 0] = \mathbf{0}.$

Arbitrariness of Potentials ϕ and A

Potentials ϕ and ***A*** *are not defined uniquely*; *i.e., many different potentials lead to the same intensitiesoftheelectricandmagneticfields*.Ifwemadeinϕ and***A***thefollowingmodifications:

$$\phi = \phi - \frac{1}{c}\frac{\partial f}{\partial t} \tag{G.9}$$

$$\boldsymbol{A} = \boldsymbol{A} + \nabla f, \tag{G.10}$$

where f is an *arbitrary differentiable function* (of x, y, z, t), then ϕ and ***A*** lead to *the same* E and ***H***:

$$E = -\nabla\phi' - \frac{1}{c}\frac{\partial \boldsymbol{A}'}{\partial t} = \left(-\nabla\phi + \frac{1}{c}\nabla\frac{\partial f}{\partial t}\right) - \frac{1}{c}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \frac{\partial}{\partial t}(\nabla f)\right) = E$$

$$\boldsymbol{H} = \nabla \times \boldsymbol{A}' = \nabla \times \boldsymbol{A} + \nabla \times \nabla f = \boldsymbol{H}.$$

Choice of Potentials A and ϕ for a Uniform Magnetic Field

From the second Maxwell equation [Eq. (G.6)], one can see that if the magnetic field ***H*** is time-independent, then we get the time-independent ***A***. Profiting from the non-uniqueness of ***A***, we choose it in such a way as to satisfy (the so-called *Coulombic gauge*)[2]

$$\nabla \cdot \boldsymbol{A} = 0, \tag{G.11}$$

which diminishes the arbitrariness but does not remove it.

Let us take an example of an atom in a uniform magnetic field ***H***. We locate the origin of the coordinate system on the nucleus, the choice being quite natural for an atom, and construct the vector potential at position $\boldsymbol{r} = (x, y, z)$ as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}[\boldsymbol{H} \times \boldsymbol{r}]. \tag{G.12}$$

As has been shown above, this is not a unique choice; there is an infinity of them. All the choices are equivalent from the mathematical and physical point of view, they differ however

[2] The Coulombic gauge, even if it is only one of the possibilities, is almost exclusively used in molecular physics. The word *gauge* comes from the railway technology of measuring different rail widths.

the economy of computations. It appears that the choice of $\boldsymbol{A}$ is at least logical. The choice is also consistent with the Coulombic gauge [see Eq. (G.11)], because

$$\nabla \cdot \boldsymbol{A} = \frac{1}{2}\nabla \cdot [\boldsymbol{H} \times \boldsymbol{r}] = \frac{1}{2}\nabla \cdot [- \boldsymbol{H} \times \boldsymbol{r}] = \frac{1}{2}\nabla \cdot [H_y z - yH_z, H_z x - zH_x, H_x y - xH_y]$$

$$\nabla \times \boldsymbol{A} = \frac{1}{2}\nabla \times [\boldsymbol{H} \times \boldsymbol{r}] = \frac{1}{2}\nabla \cdot [\boldsymbol{H} \times \boldsymbol{r}] = \frac{1}{2}\nabla \times [H_y z - yH_z, H_z x - zH_x, H_x y - xH_y]$$
$$= \frac{1}{2}\left[\frac{\partial}{\partial x}(H_y z - yH_z) + \frac{\partial}{\partial y}(H_z x - zH_x) + \frac{\partial}{\partial z}(H_x y - xH_y)\right] = 0,$$

and also with the Maxwell equations [see Eq. (G.6)], because

$$= \frac{1}{2}\left[\frac{\partial}{\partial y}(H_x y - xH_y) - \frac{\partial}{\partial z}(H_z x - zH_x), \frac{\partial}{\partial z}(H_y z - yH_z) - \frac{\partial}{\partial x}(H_x y - xH_y), \frac{\partial}{\partial x}(H_z x - zH_x) - \frac{\partial}{\partial y}(H_y z - yH_z)\right] = \boldsymbol{H}.$$

Thus, this is the correct choice.

Practical Importance of this Choice

An example of possible choices of $\boldsymbol{A}$ is shown in Fig. G.1.

If we shifted the vector potential origin far from the physical system under consideration (Fig. G.1b), then the values of $|\boldsymbol{A}|$ on all the particles of the system would be giant. $\boldsymbol{A}$ would be practically uniform *within* the atom or molecule. If we calculated $\nabla \times \boldsymbol{A} = \boldsymbol{H}$ on a particle of the system, we would obtain almost $\mathbf{0}$, because $\nabla \times \boldsymbol{A}$ means the differentiation of $\boldsymbol{A}$, and for a uniform field, this yields zero. Thus, we are going to study the system in magnetic field, but the field disappeared. A very high accuracy would be needed in order to calculate correctly $\nabla \times \boldsymbol{A}$ as the differences of two large numbers, which numerically is always a risky business due to the cancellation of accuracies. *It is seen, therefore, that the numerical results depend critically on the choice of the origin of* $\boldsymbol{A}$ (which is arbitrary from the point of view of mathematics and physics). *It is always better to have the origin inside the system.*

Vector Potential Causes the Wave Function to Change Phase

The Schrödinger equation for a particle of mass m and charge q reads as

$$-\frac{\hbar^2}{2m}\Delta\Psi(\boldsymbol{r}) + V\Psi = E\Psi(\boldsymbol{r}),$$

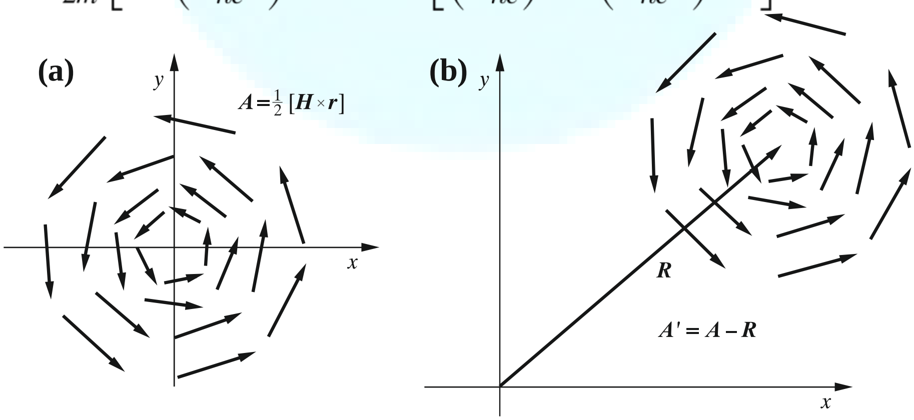
where $V = q\phi$, with ϕ standing for the scalar electric potential.

The probability density of finding the particle at a given position depends on | | rather than itself. This means that the wave function could be harmlessly multiplied by a phase factor $\Psi'(\boldsymbol{r}) = \Psi(\boldsymbol{r}) \exp[-\frac{iq}{\hbar c}\chi(\boldsymbol{r})]$, where $\chi(\boldsymbol{r})$ could be any (smooth[3]) function of the particle's position $\boldsymbol{r}$. Then, we have $|\Psi'| = |\Psi|$ at any $\boldsymbol{r}$. If $\Psi'(\boldsymbol{r})$ is as good as Ψ is, it would be nice if it satisfied the Schrödinger equation the same way that Ψ does, of course with the same eigenvalue:

$$-\frac{\hbar^2}{2m}\Delta\Psi'(\boldsymbol{r}) + V\Psi'(\boldsymbol{r}) = E\Psi'(\boldsymbol{r}).$$

Let us see what profound consequences this has. The left side of the last equation can be transformed as follows:

$$\begin{aligned}
&-\frac{\hbar^2}{2m}\Delta\Psi'(\boldsymbol{r}) + V\Psi'(\boldsymbol{r}) \\
&= -\frac{\hbar^2}{2m}\left[\exp\left(-\frac{iq}{\hbar c}\chi\right)\Delta\Psi + \Psi\Delta\exp\left(-\frac{iq}{\hbar c}\chi\right) + 2(\nabla\Psi)\left(\nabla\exp\left(-\frac{iq}{\hbar c}\chi\right)\right)\right] \\
&\quad + V\exp\left(-\frac{iq}{\hbar c}\chi\right)\Psi \\
&= -\frac{\hbar^2}{2m}\left[\exp\left(-\frac{iq}{\hbar c}\chi\right)\Delta\Psi + \Psi\nabla\left[\left(-\frac{iq}{\hbar c}\right)\exp\left(-\frac{iq}{\hbar c}\chi\right)\nabla\chi\right]\right.
\end{aligned}$$



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[3] See Fig. 2.6.

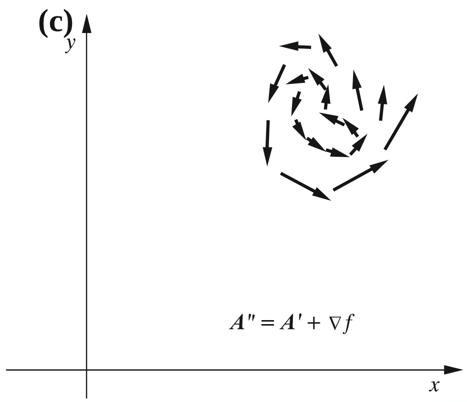

Fig. G.1. How can we understand the arbitrariness of the vector potential $\boldsymbol{A}$? Panels (a), (b), and (c) represent schematically three physically equivalent vector potentials $\boldsymbol{A}$. Panel (a) shows a section by the plane $z = 0$ (axis z protrudes toward the reader from the xy plane) of the vector field $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{H} \times \boldsymbol{r})$ with $\boldsymbol{H} = (0, 0, H)$ and $H > 0$. It is seen that the vectors $\boldsymbol{A}$ become longer and longer, when we are going out of the origin (where $\boldsymbol{A} = \boldsymbol{0}$), they "*rotate*" counterclockwise. Such $\boldsymbol{A}$ determines that $\boldsymbol{H}$ is directed perpendicularly to the page and oriented toward the reader. By the way, note that any *shift* of the obtained potential should give the same magnetic field orthogonal to the drawing. This is what we get (b) after adding, according to Eq. (G.10), the gradient of function $f = ax + by + c$ to potential $\boldsymbol{A}$, because $\boldsymbol{A} + \nabla f = \boldsymbol{A} + (\boldsymbol{i}a + \boldsymbol{j}b) = \boldsymbol{A} - \boldsymbol{R} = \boldsymbol{A}$, where $\boldsymbol{R} = -(\boldsymbol{i}a + \boldsymbol{j}b) = \boldsymbol{const}$. The transformation is only one of possible transformations. If we took an arbitrary smooth function $f(x, y)$ e.g., with many maxima, minima and saddle points (as in the mountains), we would deform (b) by expanding or shrinking it like a pancake. In this way we might obtain the situation shown on (c). All these situations are physically indistinguishable (if the scalar potential ϕ is changed appropriately).

$$+ 2(\nabla\Psi)\left[\left(-\frac{iq}{\hbar c}\right)\exp\left(-\frac{iq}{\hbar c}\chi\right)\nabla\chi\right]]$$

$$+ V\exp\left(-\frac{iq}{\hbar c}\chi\right)\Psi$$

$$= -\frac{\hbar^2}{2m}\left[\exp\left(-\frac{iq}{\hbar c}\chi\right)\Delta\Psi + \Psi\left(-\frac{iq}{\hbar c}\right)\right.$$

$$\left.\left[\left(-\frac{iq}{\hbar c}\right)\exp\left(-\frac{iq}{\hbar c}\chi\right)(\nabla\chi)^2 + \exp\left(-\frac{iq}{\hbar c}\chi\right)\Delta\chi\right]\right]$$

$$-\frac{\hbar^2}{2m}2(\nabla\Psi)\left[\left(-\frac{iq}{\hbar c}\right)\exp\left(-\frac{iq}{\hbar c}\chi\right)\nabla\chi\right] + V\exp\left(-\frac{iq}{\hbar c}\chi\right)\Psi.$$

Dividing the Schrödinger equation by $\exp^{\left(-\frac{iq}{\hbar c}\chi\right)}$, we obtain

$$-\frac{\hbar^2}{2m}\left[\Delta\Psi + \Psi\left(-\frac{iq}{\hbar c}\right)\left[\left(-\frac{iq}{\hbar c}\right)(\nabla\chi)^2 + \Delta\chi\right]\right.$$

$$\left. + 2(\nabla\Psi)\left[\left(-\frac{iq}{\hbar c}\right)\nabla\chi\right]\right] + V\Psi = E\Psi(\boldsymbol{r}).$$

Let us define a vector field $\boldsymbol{A}(\boldsymbol{r})$ by using function $\chi(\boldsymbol{r})$:

$$A(r) = \nabla\chi(r). \qquad \text{(G.13)}$$

Hence, we have

$$-\frac{\hbar^2}{2m}\left[\Delta\Psi + \Psi\left(-\frac{iq}{\hbar c}\right)\left[\left(-\frac{iq}{\hbar c}\right)A^2 + \nabla A\right] + 2(\nabla\Psi)\left[\left(-\frac{iq}{\hbar c}\right)A\right]\right] + V\Psi = E\Psi(r),$$

and introducing the momentum operator $\hat{p} = -i\hbar\nabla$, we obtain

$$\frac{1}{2m}\left[\hat{p}^2\Psi + \Psi\left[\left(\frac{q}{c}\right)^2 A^2 - \left(\frac{q}{c}\right)\hat{p}A\right] - 2(\hat{p}\Psi)\left(\frac{q}{c}\right)A\right] + V\Psi = E\Psi(r),$$

or, finally,

$$\frac{1}{2m}\left(\hat{p} - \frac{q}{c}A\right)^2\Psi + V\Psi = E, \qquad \text{(G.14)}$$

which is the equation corresponding to the particle moving in electromagnetic field with vector potential A; see p. 762.

Indeed, the last equation can be transformed in the following way:

$$\frac{1}{2m}[\hat{p}^2\Psi + \left(\frac{q}{c}\right)^2 A^2\Psi - \frac{q}{c}\hat{p}(A\Psi) - \frac{q}{c}A\hat{p}\Psi] + V\Psi = E,$$

which, after using the equality[4] $\hat{p}(A\Psi) = \Psi\hat{p}A + A\hat{p}\Psi$, gives the expected result, Eq. (G.14).

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[4] Remember that $\hat{p}$ is proportional to the first derivative operator.

Therefore, note the following:

- fromthefactthatawavefunction (with*acertain*phase)satisfiestheSchrödingerequation,
- requiring that a change of the phase coming from the multiplication of by exp $[-\frac{iq}{\hbar c}\chi(\mathbf{r})]$ does not prevent one from satisfying the Schrödinger equation by $\Psi\exp[-\frac{iq}{\hbar c}\chi(\mathbf{r})]$ (in the

text,$\chi(\mathbf{r})$willbecalledthe*phase*,althoughinreality,thephasechangeisequalto $-\frac{q}{\hbar c}\chi(\mathbf{r})$), onereceivestheSchrödingerequationfor withoutthisadditionalchangeofthephase.This equationcontainsamodifiedmomentum(andthereforealsowiththemodifiedHamiltonian) of the particle[4] by the so-called vector potential $\mathbf{A}(\mathbf{r})$: $\hat{\boldsymbol{p}} \rightarrow (\hat{\boldsymbol{p}} - \frac{q}{c}\mathbf{A})$.

Hence, we have a link between the appearance of the phase (or rather its change χ) with the appearance of the vector potential **A** in the Schrödinger equation. This makes us believe that we got a more general form of the Schrödinger equation (this with **A**).

A function with phase $\chi_1(\mathbf{r})$ corresponds to the vector potential $\mathbf{A}_1(\mathbf{r})$, while the function with the phase $\chi_1(\mathbf{r})+\chi(\mathbf{r})$ corresponds to $\mathbf{A}_2(\mathbf{r})$, where $\chi \equiv \mathbf{A}_2(\mathbf{r})-\mathbf{A}_1(\mathbf{r})$. Therefore, thesetwovectorpotentialsdifferbythegradientofafunction,andthisisallowed[according to Eq. (G.10)] without having to modify any physical phenomena.

Both functions, this with the phase χ_1 as well as that with the phase $\chi_1(\mathbf{r}) + \chi(\mathbf{r})$, give, of course, the same probability density.[5] Therefore, we are free to change $\mathbf{A}_1(\mathbf{r})$ to $\mathbf{A}_2(\mathbf{r})$, provided that we compensate the new choice by changing the phase χ according to $\nabla\chi \equiv \mathbf{A}_2(\mathbf{r}) - \mathbf{A}_1(\mathbf{r})$

.

Notethatthephysicallyequivalentvectorpotentialsalsomaycontainavectorfieldcomponent **A** (the same in both cases), *which is not a gradient (of any function)*. For example, it contains a vortexlike field, which is not equivalent to any gradient field. The curl of such a field is nonzero, while the curl of a gradient of any function does equal zero. Thus, the **A** itself may contain an unknown admixture of the gradient of a function. Hence, any experimental observation is determined solely by the non-gradient component of the field. For example, the magnetic field $\mathbf{H}_2$ for the vector potential $\mathbf{A}_2$ is

$$\mathbf{H}_2 = \text{curl}\mathbf{A}_2 \equiv \nabla\times\mathbf{A}_2 = \nabla\times(\mathbf{A}_1+\nabla\chi) = \nabla\times\mathbf{A}_1+\nabla\times\nabla\chi = \nabla\times\mathbf{A}_1+0 = \text{curl}\mathbf{A}_1 = \mathbf{H}_1,$$

[4] Note that if we put $c = \infty$ (non-relativistic approximation), no modification occurs.

[5] For any real

where both vector potentials ($\mathbf{A}_1$ and $\mathbf{A}_2$) correspond to the same physical situation because $\mathbf{H}_1 = \mathbf{H}_2$. This result is obtained, because curl$(\text{grad}\chi) \equiv \nabla \times \nabla\chi = 0$.

$$\chi_1 \text{ and } \chi, \left|\exp[-\tfrac{iq}{\hbar c}\chi(\mathbf{r})]\right| = 1 \text{ and } \left|\exp[-\tfrac{iq}{\hbar c}(\chi_1(\mathbf{r}) + \chi(\mathbf{r}))]\right| = 1.$$

If a particle moves in a vector potential field $\boldsymbol{A}$ from $\boldsymbol{r}_0$ to $\boldsymbol{r}$, then its wave function changes the phase by δ:

$$\delta = -\frac{q}{\hbar c}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{A}(\boldsymbol{r})\mathrm{d}\boldsymbol{r}.$$

Puttingitinadifferentway,ifthewavefunctionundergoesaphasechange,thentheparticle moves in a vector potential of an electromagnetic field.

The Incredible Aharonov-Bohm Effect

In a small domain (say, in the center of the Brussels marketplace, where we like to locate the originofthecoordinatesystem),thereisamagneticfieldfluxcorrespondingtothefieldintensity $\boldsymbol{H}$ directed along the z-axis (perpendicular to the marketplace surface). Now, let us imagine a particle of the electric charge q enclosed in a 3-D box (say, a cube) of small dimensions located at a *very large distance* from the origin, and therefore from the magnetic flux–say, in Lisbon. Therefore, the *magnetic field in the box equals zero*. Now, we decide to travel with the box: from Lisbon to Cairo, Ankara, St. Petersburg, Stockholm, Paris, and then back to Lisbon. Did the wave function of the particle in the box change during the journey?

Let us see. The magnetic field $\boldsymbol{H}$ is related to the vector potential $\boldsymbol{A}$ through the relation $\nabla \times \boldsymbol{A} = \boldsymbol{H}$. This means that the particle was all the time subject to a huge vector potential field (see Fig. G.1), although the magnetic field was practically zero. Since the box has gone back to Lisbon, then the phase acquired by the particle in the box[6] is an integral over the closed trajectory (loop):

$$\delta = -\frac{q}{\hbar c}\oint \boldsymbol{A}(\boldsymbol{r})\mathrm{d}\boldsymbol{r}.$$

However, from the Stokes equation, we can replace the integral by an integral over surface enclosed by the loop

$$\iint \mathbf{n}\cdot \text{curl}\mathbf{A}\,\mathrm{d}S,$$

[6] A nonzero δ requires a more general $\boldsymbol{A}$ than the one satisfying Eq. (G.13).

where **n** is a unit vector normal to the surface, and d*S* is an infinitesimal element of the surface that is enclosed in the contour. We can also write this as *qq* **H** d*S* __ ,

$$\delta \quad - \iint \mathbf{n} = -\frac{}{\hbar c} \quad := -\frac{c}{\hbar}$$

where represents the *magnetic flux* (of the magnetic field ***H***) intersecting the loop surface, which contains in particular the famous marketplace of Brussels. Thus, despite the fact that the particle could not feel the magnetic field ***H*** (because it was zero in the box), its wave function underwent a change of phase, which is detectable experimentally (in interference experiments).

Does the pair of potentials ***A*** and ϕ contain the same information as E and ***H*** do? The Aharonov-Bohm effect (see also p. 901) suggests that the most important elements are ***A*** and ϕ!