METRIC SPACES

Euclidian Spaces R^n

$\Downarrow$

Metric spaces – A set with a distance function

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Topological spaces – A set with neighbourhood system

DEFINITION

Metric on a Non-empty Set: Let X be a non empty set. A metric d on X is a function

$d: X \times X \rightarrow [0, \infty)$ satisfying the following condition
(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$.

- If d is a metric on X, then the algebraic structure (X, d) is called a metric space.

EXAMPLE

- The Usual or Euclidean metric on R is given by,

  d: R$\times R \rightarrow [0, \infty)$ defined by $d(x, y) = |x - y|$

- The plane R^2 with the "usual distance" (measured using Pythagoras's theorem) is defined by

  d: $R^2 \times R^2 \rightarrow [0, \infty)$

  d$((x_1, y_1), (x_2, y_2))$ = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

- The plane with the supremum or maximum metric

  d: $R^2 \times R^2 \rightarrow [0, \infty)$ defined by

$$d((x_1, y_1), (x_2, y_2)) = max(|x_1 - x_2|, |y_1 - y_2|).$$

  It is often called the infinity metric d_∞.

- Let $X \neq \emptyset$. Then the function $d: X \times X \rightarrow [0, \infty)$ defined by

$$d(x, y) = \{1, x \neq y \; 0, x = y$$

  is a metric on X, called the Discrete matric.

- X= C[a,b] over R then

$$d(f,g) = \int \quad |f(x) - g(x)|dx$$

is a metric on C[a,b].

PSEUDO METRIC

$Let\ X\ be\ a\ non\ empty\ set,\ a\ pseudo$ metric d on X is a function from $X \times X \to [0, \infty)$ satisfying the following conditions:

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$

EXAMPLE

- d: R × R to R is a function defined by
  d (x , y) = $|x^2 - y^2|$ is a pseudo metric on R
- d$(x, y) = min\{|x|, |y|\}$ is a pseudo metric on R

BALLS IN A METRIC SPACE

Let (X, d) be a metric space, $r \in R$ be a positive real number and let $x \in X$.

OPEN BALL

An open ball centered at x with radius r is defined by $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

Ex:

- $(-1,1)$ is an open ball in R with usual metric centered at 0 and radius 1.

CLOSED BALL

A closed ball centred at x with radius r is defined by

$$B^{-}{}_{d}(x, r) = \{y \in X: d(x, y) \leq r\}.$$

Ex:

- [-1,1] is a closed ball in R with usual matric centered at 0 and radius 1

NEIGHBOURHOOD

Let X be a metric space with metric d, let p ∈ X. A set N ⊂ X is called a neighborhood of p if it contains some open ball with center p. In other words, N is a neighborhood of p if there exists r $\in R^+$ such that $B_d(p, r) \subset$ N.

BOUNDARY OF A BALL

Boundary of a ball centered at x with radius r is given by

$$\partial B_d(x, r) = \{y \in X: d(x, y) = r\}.$$

Ex:

- {-1,1} is the boundary of a ball in R with usual matric centered at 0 and radius 1.

EXTERIOR OF A BALL

Exterior of a ball centred at x with radius r is given by

Ext($B_d(x,r)$)= $\{y \in X: d(x,y) > r\}$.

Ex:

- $(-\infty, -1) \cup (1, \infty)$ is the exterior of the open ball (-1, 1).

SUPREMUM OF SETS

Supremum of A denoted by sup A is defined as the smallest upper bound of A. That is it is the smallest real number y such that a $\leq$ y for every a in A.

- A non empty set of real numbers which has no upper bound, therefore no least upper bound in R, Denote it as

  Sup A = +∞

  If A is an empty set, we put

  Sup A = -∞

INFIMUM OF SETS

If A is nonempty and has a lower bound, $inf\ A$ is the largest real number x such that x $\leq a$ for every a in A;

- If A is non empty and has no lower bound, we put

  $Inf\ A$ = -∞

- If A is an empty set, we put

  $Inf\ A$ = +∞

DIAMETER OF A SET

Let $(X$,d) be a metric space and let $A \subseteq X$. Then the diameter of A is defined as $diam\ (A) = sup\{d(x,y): x, y \in A\}$.

- If supremum does not exist, then $diam(A) = \infty$.

EXAMPLE

- 1 is the diameter of the set [0,1] in R with usual metric.
- Singleton set has diameter zero.

BOUNDED SET

Let (X, d) be a metric space and let $A \subseteq X$. A is said to be a bounded set in X if $diam\ (A) < \infty$.

EXAMPLE

[0,1] is a bounded set in R with usual metric

DISTANCE FROM A POINT TO A SET

Let (X, d) be a metric space and let $A \subseteq X$. For $x \in X$, the distance between x and A is defined as $dist\ (x, A) = inf\{d(x, y): y \in A\}$.

$dist\ (x, \emptyset) = \infty$.

DISTANCE BETWEEN TWO SETS

Let (X, d) be a metric space and let A, B $\subseteq X$. The distance between A and B is defined as $dist\ (A, B) = inf\ \{d(x, y): x \in A, y \in B\}$.

- $dist\ (A, \emptyset) = \infty$.

LIMIT POINT OF A SET

Let (X, d) be a metric space and let $A \subseteq X$. An element $x \in X$ is said to be a limit point of A if $dist\ (x, A \setminus \{x\}) = 0$

EXAMPLE

Let $x \in R$, then $(x, Q \setminus \{x\})\ = 0$ in R with usual metric. Hence any real number is a limit point of Q.

ISOLATED POINT OF A SET

Let (X, d) be a metric space and let $A \subseteq X$. An element $x \in A$ is said to be an isolated point of A if $dist\ (x, A\ \setminus \{x\}) > 0$

EXAMPLE

- 2 is the isolated point of the set {2}∪ $(0,1)$.

DISCRETE SET

Let $\langle X, d)$ be a metric space and let $A \subseteq X$. A is said to be a discrete set in X if all of its points are isolated.

EXAMPLE

- {1,2,3} is a discrete set in R

BOUNDARY POINT OF A SET

Let (X, d) be a metric space and $A \subseteq X$. An element x∈ X is said to be a boundary point of A if

$$dist(x, A) = dist(x, A^c)\ = 0.$$

BOUNDARY OF A SET

The set of all boundary points of A is called the boundary of a set denoted by ∂A.

EXAMPLE

- $\partial\emptyset = \emptyset$
- $\partial R = \emptyset$
- $\partial Q = R$

CLOSURE

Let (X, d) be a metric space and A ⊆ X. The closure of A is defined by $cl(A) = \underline{A} = A \cup \partial A$. In other words $\underline{A} = \{x \in X\colon dist(x, A)$=0}.

EXAMPLE

- $\underline{N}$=N
- $\underline{Q}$=R
- Closure of a finite set is itself.
- $\underline{R}$=R
- $\underline{C}$=C

INTERIOR OF A SET

Let (X, d) be a metric space and let $A \subseteq X$. The interior of A is defined as A^0 ={x∈ $X\colon dist(x, A^C) > 0\}$

Notations: $int\ (A)$, A°

- Any element of A° is called an interior point of A.
- Interior of A is the largest open set contained in A
- $int\ A$ is equal to the union of all open subsets of X contained in A.
- A open in X ⇒ $int\ A = A$

EXAMPLE

- [0, 1] has its interior (0, 1).

EXTERIOR OF A SET

Let (X, d) be a metric space and let $A \subseteq X$. The exterior of A is defined by ext $\{A\} = \{x \in X$: dist $(x, A) > 0\} = (A^{-})^c$.

EXAMPLE

- $ext(Q)$=∅
- $ext(N) = N^c$
- $ext(R) = \emptyset$
- $ext(\emptyset) = R$

OPEN SET

Let (X, d) be a metric space and let $A \subseteq X$. A is said to be an open set in X if all points of A are interior points of A.

- For any subset A of X, int $\{A\}$ is always an open set in X.

RESULTS

- Let (X, d) be a metric space with A$\subseteq X$. A is said to be open in X iff $int(A) = A$.
- Let (X, d) be a metric space. Then
  - (a) Arbitrary union of open set is open.
  - (b) Finite intersection of open set is open.

CLOSED SETS

Let (X, d) be a metric space and let A$\subseteq X. A$ is said to be closed set in X if all limit points of A are in A.

For any subset A of X $cl(A)$ is a closed set in X.

EXAMPLES

- In R with usual metric $\emptyset, R, N$ are closed sets.

RESULTS

- Let (X, d) be a metric space and A$\subseteq X$. A is closed in iff $\underline{A} = A$.
- Let (X, d) be a metric space. Then,
  - (a) Arbitrary intersection of closed set is closed.
  - (b) Finite union of closed set is closed.

DENSE SETS

Let (X, d) be a metric space and let A$\subseteq X$. A is dense in X if

$dist(x, A) = 0 \ \forall \ x \in X.$

EXAMPLE

- Q is dense in R with usual metric.

RESULTS

Let (X, d) be a metric space and let A$\subseteq X$ A is dense in X iff

(a) $cl(A) = X$

(b) $A \cap V \neq \emptyset$ for any nonempty open set $V \subseteq X$.

NOWHERE DENSE SET

Let (X, d) be a metric space. And let A⊆ X A is nowhere dense in X if $(\underline{A})^0 = \emptyset$

If A is nowhere dense in X, then A^C is dense in X. Converse is not true.

SEPERABLE SPACE

Let (X, d) be a metric space. X is said to be separable space, if it has a countable dense subset.

BOUNDED METRIC

Let (X, d) be a metric spaces called bounded metric if $d(x, y) < k$ for some $k > 0$ (real) $\forall x, y \in X$.

EQUIVALENT METRICS

Let d_1 and d_2 be two metric on a set X is called equivalent metric if
(i) Given any d_1-open sphere S_{d_1} with center x, $\exists$ a d_2-open sphere S_{d_2} with center x such that $S_{d_1} \subseteq S_{d_2}$.
(ii) Given any d_2-open sphere S_{d_2} with center x, $\exists$ a d_1-open sphere S_{d_1} with center x such that $S_{d_2} \subseteq S_{d_1}$.

SEQUENCE IN METRIC SPACES

Let (X, d) be a metric space.
Let $f: N \to X$ be a function then
$(f(1), f(2), f(3), \dots)$ is called sequence in X.

CONVERGENCE

Let $< Xn >$ be a sequence in metric space (X, d). $< Xn >$ is said to converge to a point $x \in X$ if for every $\varepsilon > 0$, $\exists\, n_0 \in$ N such that

$\forall n \geq n_0$ we have $d(x_n, x) < \epsilon$.

We denote it as $x_n \to x$.

CAUCHY SEQUENCE

$< Xn >$ is said to be a Cauchy sequence in a metric space (X, d)

if $\forall \varepsilon > 0, \exists\, n_0 \in N$ such that $\forall n, m \geq n_0$ we have d(x_n, x_m)$< \epsilon$.

Every convergent sequences are Cauchy but the converse need not be true.

COMPLETE METRIC SPACE

Let (X, d) be a metric space, (X, d) is called complete metric space if every Cauchy sequence in (X, d) converges in (X, d).

Example:

(a) R with usual metric is complete.

(b) (0,1) with usual metric is not complete

(c) Discrete metric spaces are complete

BOUNDED SEQUENCE

Let (X, d) be a metric space, a sequence (x_n) in X is bounded if $\{x_n : n \in N\}$ is a bounded set.

EVENTUALLY CONSTANT

Let (X, d) be a metric space, a sequence (x_n) is eventually constant if there are $x \in X$ and $n_0 \in N$ such that $x_n = x \; \forall \; n \geq n_0$

RESULTS

In complete metric space, Cauchy ⇔ convergent

Discrete metric is complete. Therefore Cauchy ⇔ convergent

TOPOLOGICAL SPACES

Definition:

A topology on a set X is a collection T of subsets of X having the following properties.
(i) $\phi, X \in T$
(ii) $G_i \in T; i \in \Delta \Rightarrow \cup_{i \in \Delta} G_i \in T$
where Δ is an index set.
(iii) $G_i \in T: i = ,2, \dots, n \Rightarrow \cup_{i=1}^{n} G \in T$
Then (X, T) is called topological space.

Open set

Let (X, T) be a topological space. A subset A of X is said to be open in (X, T) if A$\in T$.

Open Neighbourhoods

$x \in X, N_x = \{U \in T : x \in U\}$

Members of N_x are called open neighbourhood of x on (X, T).

Closed set

A subset A of (X, T) is said to be a closed set in X if (X\A) is open set.

- Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric.

Examples:

- Trivial topology or Indiscrete topology on X is given by T= $\{X, \emptyset\}$.
- Discrete topology or power set topology $T = P(X)$
  X be a non-empty set $T = P(X)$ is a topology called discrete topology, $X, P(X))$
- In power set topology any subset of X is open in X.
  It is a topology in which every singleton set is open, since their complement belongs to the same set. Hence it is closed.

Cofinite/finite complement topology

$T = \{A \subseteq X: A^c = X$ or A^c is finite $\} = T_{cf}$

If X is a finite set, cofinite topology becomes discrete topology.

Cocountable Topology

$$T = \{A \subseteq X: A^c = X \text{ or } A^c \text{ is countable}\} = T_{cc}$$

If X is a Countable set, then cocountable topology becomes discrete topology.

Sierpinski Topology

Let X=$\{a, b\}$, then T= $\{\emptyset, X, \{a\}\}$ is a topology on X *called Sierpinski*

Topology.

Euclidian Topology

(R^n, d_E) is the Euclidean metric space.

(R^n, T_E) is the Euclidian topology which is generated by all open balls and also by open boxes.
i.e $\prod_{j=1}^{n} \quad (a_j, b_j)$

Upper limit topology

$$T = \{(a, b]: a < b \text{ are reals}\} = T_{up}$$

Lower limit topology

$$T = \{[a, b): a < b \text{ are reals}\} = T_{low}$$

$X = \{a, b\}$, topologies on, X are given by

- $\{X, \emptyset\}$,
- $\{X, \emptyset, \{a\}\}$,
- $\{X, \emptyset, \{b\}\}$,
- $\{X, \emptyset, \{a\}, \{b\}\}$

Interior point

Let (X,T) be a topological space $A \subseteq X, x \in X$ is a called interior point of A if $\exists G \in T$ with x such that $G \subseteq A$.

Collection of all interior point of A is called interior of the set denoted by A° or $Int\ (A)$

Limit point

Let (X,T) be a topological space and $A \subseteq X, x \in X$ is called limit point of A if $\forall G \in T$ with x s.t. $\{G - \{x\}\} \cap A \neq \phi$

Collection of all limit point of A is called derived set, denoted by $D(A)$ or A'.

Adherent point

Let (X,T) be a topologic space and $A \subseteq X, \alpha \in X$ is called Adherent point of A,
if $\forall G \in T$ with x s.t. $G \cap A \neq \phi$.

Boundary point

Let (X,T) be a topological space and A⊆ X $x \in X$, is a boundary point of A, if $x\ is$ neither interior of A nor interior of A^C.

Collection of all boundary points of A is called boundary set of A denoted by $\partial(A)$.

- $\partial(A) = X - \{A^o \cup (X-A)^o\}$
- $\partial(A)$ is closed set

Interior of a set

Let (X,T) be a topological space. Let $A \subseteq X$, then interior of A denoted by A^0 and define as :

$$A^0 = \cup \{G: G\ is\ open\ in\ X; G \subseteq A\}$$

- A^0 is an open set
- A is open iff A^0 =A
- $(A^0)^0$=A

Closure of set

Let (X,T) be a topological space and $A \subseteq X$ then closure of A is denoted by A^- and is defined as

$$A^- = \cap \{F \mid F \text{ is closed in } X: F \supseteq A\}$$

- A^- is closed set
- $\underline{A^-} = A^-$

- A is closed set iff $A^{-} = A$

Dense set

Let (X,T) be a topological space and A $\subseteq X$.

A is dense in itself if A$\subseteq D(A)$.

A is said to be dense or everywhere dense in X if $A^{-} = X$.

Somewhere Dense

A is said to be some where dense if $(\underline{A})^{0} \neq \emptyset$; that is closure of A contains some open sets.

Nowhere Dense

A is said to be nowhere dense if it is not somewhere dense.

i.e A is said to be nowhere dense or non dense in X if

$(\underline{A})^{0}$=∅ .

Separable space

A topological space is called separable if it has a countable dense subset.

Relative Topology/ Subspace Topology

Let (X, T) be a topological spaces $Y \subseteq X$ now we define $U = \{G \cap Y \mid G$ is open in X or $G \in T\}$.

Then U is a topology on Y and called Relative Topology.

- A is open in Y. Iff $\exists G$ open in X such that $A = G \cap Y$.
- A is closed in Y iff $\exists F$ closed in Y such that $A = F \cap Y$.
- A is open in Y, Y is open in X.Then A is open in X.
- A is closed in Y and Y is closed in X. Then A is closed in X.

Neighborhood of a Point

Let (X, T) be a topological space $A \subseteq X$ is called neighborhood of a point $x \in X$ if $\exists G \in T$ with $x \in G$ such that $G \subseteq A$.

- Any open set $G \subseteq X$ with $x \in G$ is also nbd of a point $x \in X$.
- If A is a neighborhood of a point $x \in X$, then $A - \{x\}$ is called deleted neighborhood of x.

COMPACTNESS

Cover

Let (X, T) be a topological space. Let $G = \{A | A \subseteq X\} \subseteq P(X)$ and

B$\subseteq X$, B $\subset \cup_{A \in G} A$. Then G is called cover of B.

Open cover

If $G \subset T$ then G is called open cover of B

Compact Set

Let (X, T) be a topological space and $A \subset X$ is said to be compact if each cover reducible to finite sub covering.

Heine Borel Theorem

A closed bounded subset of R is called compact.

Locally Compact

A topological space (X, T) is said to be locally compact if each element $x \in X$ has a compact neighbourhood.

- Compact ⇒ locally compact converse need not be true.

Sequentially compact

Let (X, d) be a metric space is said to be a sequentially compact if every sequence in X has a convergent subsequence.

COMPACTIFICATION

Enlarging a given space into a compact Hausdorff space

Hausdorff spaces

Is a topological space where for any two distinct points there exist neighborhoods of each which are disjoint from each other.

- (X, T) be a topological space which is not compact. By a compactification of X we mean a compact Hausdorff space Y containing X as a dense subset.

One point compactification

Given a topological space X, we wish to construct a compact space Y by appending one point: $Y = X \cup \backslash\{\infty\}$. This is called a one-point compactification of X.

Describing the Topology of Y

- *not containing* ∞: open subsets *U* of *X*, or
- Containing ∞: sets *Y*–*C*, where *C* is a closed subset of *X* which is compact.
- A one-point compactification of (0, 1) (or R) is given by the circle $S := \{(x, y) \in R \times R : x^2 + y^2 = 1\}$ in the plane.

- A one-point compactification of [0, 1) is given by [0, 1].
- A one-point compactification of (0, 1) ∪ (2, 3) is given by the union of two circles which are tangent to each other.

CONNECTEDNESS

Separated sets

Let A,B be subsets of a topological space (X, T).Then the sets A and B are said to be separated iff

1. A$\neq \emptyset, B \neq \emptyset$
2. A$\cap \underline{B} = \emptyset, \underline{A} \cap B = \emptyset$

Disconnectedness

- Let (X, T) be a topological space and

$A \subseteq X$. The set A is said to be disconnected or

separated subset of X if $\exists$ G, H $\in T$ such that

1.$A \cap G, A \cap H \neq \emptyset$

2.$(A \cap G) \cap (A \cap H) \neq \emptyset$

3.$A = (A \cap G) \cup (A \cap H)$

Connectedness

- Two point a and b of a topological space (X,T) are said to be connected iff they are contained in a connected subset of X.
- X is said to be connected iff it is not disconnected.

Path

Let (X, T) be a topological space. Let I be the set of all real numbers belonging to the closed interval $[0,1]$ with usual topology.

Let a, b $\in$ X be arbitrary.

A continuous map $f: I \rightarrow X$ with the properties $f(0) = a, f(1) = b,$

is called a path from a point a to the point b.

Path Connected

- A topological space (X, T) is said to be path connected if

$\forall a, b \in X$ there is a path joining a and b.

- Path connected ⇒ connected
- Connected ⇏ path connected

Example:

Topological sine curve

Basis

If X is a set, a basis for a topology on X is a subset β of $P(X)$ such that

1. every open set is a union of members of β or $\bigcup_{B\in\beta} \in B = X$
2. $x \in B_1 \cap B_2, B_1, B_2 \in \beta \Rightarrow$ there exist a $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Local Basis

Let (X, T) be a topological space and $x \in X$ let β_x be the collection of open sets containing x. Then β_x is said to be a local basis at x if

1. If B is an open set containing x then there exist an element $B_x \in \beta_x$ such that $x \in B_x \subseteq B$

SubBase

(X, T) be a topological space, $S \subseteq T$ is a subbasis if collection of all finite intersection is a basis of T.

First countable

A topological space X is said to be first countable if it has a countable local basis at each point $x \in X$

Second countable:

A topological space X is said to be second countable if it has a countable open basis.

Result

Second countable ⇒ first countable

Lindelof Space

Every open cover has countable sub cover.

- Every second countable space is Lindelof.
- Product of two Lindelof space need not be Lindelof.

Box Topology

$\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces. Let $X = \prod_{\alpha\in J} X_\alpha$

$$\beta = \{\prod_{\alpha\in J} U_\alpha : U_\alpha \text{ is open in } X\}$$

The topology induced by β is called box topology.

Product topology

$\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces. $X = \prod_{\alpha\in J} X_\alpha$

$\hat{\beta} = \{\prod_{\alpha\in J} U_\alpha : U_\alpha = X_\alpha \text{ except for a finite values of } \alpha\}$

Then $\hat{\beta}$ forms a basis for a topology called product topology.

- Box topology is finer than product topology
- $X = \prod_{\alpha\in J} X_\alpha$ is compact w.r.t product topology iff each X_α is compact.
- $X = \prod_{\alpha\in J} X_\alpha$ is connected w.r.t product topology iff each X_α is connected.
- Closed subspace of compact space is compact.
- Closed subspace of Lindelof space is Lindelof.

Components

The largest connected subspace of a topological space is called component.

- Component partitions the space.
- Components are always closed**.**
- Connected set need not be components.
- A connected set which is both open and closed ⇒ component.
- (X, T) be a topological space,
  A connected ⇒ A closed, need not be open
  A open and closed ⇒ A is a component.

Finite and Infinite Set

A set with finite number of elements are called a finite set and A set with infinite number of elements are called an infinite set

Denumerable Set

An infinite set is said to be countably infinite/ denumerable/ enumerable if it is equivalent to the set of natural numbers $\mathbb{N}$

Countable and Uncountable Set

A set which is either empty or finite or countably infinite is called a countable set.

Otherwise it is called uncountable set

- Countable union of countable sets are countable
- Finite product of countable sets are countable

- Power set of a countably infinite set is uncountable

Cardinality of N is α_0
Cardinality of R is c
Denumerable sets have cardinality α_0

- $\alpha_0 < c$
- α_0+n= α_0
- $2^{\alpha_0} = c$
- $\alpha_0^{\alpha_0} = c$
- n.α_0= α_0