Symmetries and conservation laws

1 Introduction

Conservation laws and symmetries have always been of considerable interest in science. They are important in the formulation and investigation of many mathematical models. They were used, e.g. for proving global existence theorems [1]–[3], in problems of stability [4], [5], in elasticity for studying cracks and dislocations [6], [7], in astrophysics [8]–[10], in designing new radio antennas [11] and so on (see also [12]).

Let us look at the use of symmetries and conservation laws, e.g. in celestial mechanics. In 1609, J. Kepler formulated two important laws known as Kepler's first and second laws. His first law states that the orbit of a planet is an ellipse with the Sun as its focus. The second law says that if we join the Sun and a planet by a straight line, the line will sweep out equal areas at equal times.

What was important in these discoveries is that Kepler explained how the planets moved. The next step, the explanation why they moved in such a way, was given by I. Newton [13] in 1687. He formulated his law of gravity:

$$F = G\frac{m_1 m_2}{r^2}, \quad (1.1)$$

where F is the force of gravity between two particles, G is a gravitational constant, m_1 and m_2 are the masses of the particles and r is the distance between them. At the beginning Newton tried to use a formula with r^3 instead of r^2. However, he found out that it was not fruitful. When Newton used the force of gravity (1.1) in his second law of motion, he obtained that planets moved in ellipses. It proved to him that he was on the right track. Thus, his way of discovery was by trial and error.

P.–S. Laplace showed that planets' movement along ellipses followed from the conservation law calculated by him, i.e. the conservation law for the vector (see [14], Vol.1, Book II, Chap. III, Section 18):

$$\mathbf{A} = \mathbf{v}\times\mathbf{M} + \mu\frac{\mathbf{x}}{r}, \quad (1.2)$$

where $\mathbf{v}$ is the velocity of a planet, $\mathbf{M} = m(\mathbf{x}\times\mathbf{v})$ is the angular momentum, m is the planet's mass, $\mathbf{x}$ is a position-vector of the planet and r is the magnitude of $\mathbf{x}$. Laplace used the formal definition of a conservation law for calculation of this conserved vector.

In 1983, N. H. Ibragimov [15] showed that it was possible to calculate the vector (1.2) by using a certain symmetry of the Newton gravitational field, a Lie-Bäcklund symmetry. This symmetry is more complicated than, e.g. rotations, it depends not only on the position vector x but also on the velocity v. Thus, the idea of symmetry and the corresponding conservation law helps to explain the movement of planets in ellipses.

Kepler’s second law, the conservation of areas, follows from the conservation of angular momentum. This was established [16] independently by L. Euler and D. Bernoulli. The angular momentum corresponds to the cental symmetry of Newton’s gravitational field.

2 Conservation laws

There are several ideas for constructing conservation laws. One of them is to use the direct method, when a conservation law for a differential equation is derived by using its definition. As mentioned earlier, Laplace was the first one who used this idea in 1798.

Another idea, that certain conservation laws for differential equations obtained from a variational principle could appear from their symmetries, followed from the works of Jacobi, Klein and Noether. In 1884, Jacobi [17] showed a connection between conserved quantities and symmetries of the equations of a particle’s motion in classical mechanics. Similar result was obtained by Klein [18] for the equations of the general relativity. Klein predicted that a connection between conservation laws and symmetries could be found for any differential equation obtained from a variational principle. He suggested to Emmy Noether to investigate the possibility. She showed [19] in 1918 that the conservation laws were associated with invariance of variational integrals with respect to continuous transformation groups. Noether obtained the sufficient condition for existence of conservation laws. However, there are no explicit expressions for resulting conservation laws in Noether’s work. In 1921, following Noether’s oral remark, Bessel-Hagen [20] applied Noether’s theorem with the so-called "divergence" condition to the Maxwell equations and calculated their conservations laws.

In 1951, Hill wrote a remarkable review paper [21] where he discussed Noether’s theorem and presented the explicit formula for conservation laws in the case of a first-order Lagrangian. The formula is written in terms of variations (see [21], Eq. (43)). In 1969, inspired by Hill’s article, Ibragimov [22] proved the generalized version of Noether’s theorem. In this theorem conservations laws are related to the invariance of the extremal values of variational integrals. He derived the necessary and sufficient condition for existence of conservation laws. He also presented the explicit expressions for calculating conservation laws in the case of a Lagrangian of any order. On the basis of these theorems many conservations laws for differential equations having a Lagrangian were calculated (see collected examples in
[23]–[25]).

2.1 Concept of a conservation law

Let us consider an ordinary differential equation

$$F(t,q,\dot{q},\ddot{q}) = 0 \tag{2.1}$$

describing a motion of a dynamical system. Here t is time, $q = (q^1,...,q^s)$ are the position coordinates, $q = q(t)$, and $v = \dot{q} \equiv \frac{dq}{dt}$ is the velocity, $\ddot{q} = \frac{d^2q}{dt^2}$.

Definition 2.1. A function $C = C(t,q,v)$ is called a conserved quantity for Eq. (2.1) if

$$\frac{dC}{dt} = 0 \tag{2.2}$$

on every solution of Eq. (2.1).

In other words, the conserved quantity $C(t,q,v)$ is constant on each trajectory $q = q(t)$ and therefore is called a constant of motion. In classical mechanics Eq. (2.1) has the form

$$m\ddot{\mathbf{x}} = 0 \tag{2.3}$$

and describes a free motion of a particle with the mass m and a position vector $\mathbf{x} = (x^1,x^2,x^3)$. The equation has several conserved quantities, e.g. the energy $E = \frac{1}{2}m\mathbf{v}^2$ and the linear momentum $\mathbf{p} = m\mathbf{v}$.

Let us now consider a partial differential equation of p-th order

$$F(x,u,u_{(1)},u_{(2)},...,u_{(p)}) = 0 \tag{2.4}$$

where the function F depends on n independent variables x, $x = (x^1,...,x^n)$, m dependent variables u, $u = (u^1,...,u^m)$, and the first, second, ..., p-th order derivatives of u with respect to x denoted as $u_{(1)} = \{u^\alpha_i\}$, $u_{(2)} = \{u^\alpha_{ij}\}, \ldots, u_{(p)} = \{u^\alpha_{i_1 i_2 \ldots i_p}\}$ respectively, $\alpha = 1,...,m$ and other indices change from 1 to n.

Definition 2.2. A vector $C = (C^1,C^2,...,C^n)$ where

$$C^i = C^i(x,u,u_{(1)},...), \qquad i = 1,...,n,$$

is called a conserved vector for Eq. (2.4) if

$$\mathrm{div} C = 0 \tag{2.5}$$

on every solution of Eq. (2.4). We can also say that Eq. (2.5) is a conservation law for Eq. (2.4).

A conservation law for a system of partial differential equations can be defined similarly.

Instead of dealing with functions $u^\alpha = u^\alpha(x)$ and their derivatives, which are also functions of x, one can treat all variables, x, u and derivatives of u, as independent variables, called differential variables. Variables with the same set of subscripts will be symmetric, for example $u_{ij} = u_{ji}$ and so on. Using the idea of differential variables [26] one can reformulate the definition of a conservation law by introducing the operator of total differentiation with respect to x^i:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots + u_{ij_1\ldots j_k}^\alpha \frac{\partial}{\partial u_{j_1\ldots j_k}^\alpha} + \cdots \tag{2.6}$$

where the usual convention of summation over repeated upper and lower indices is used. Hence

$$\mathrm{div}\, C\big|_{(2.4)} \equiv D_i(C^i)\big|_{(2.4)} = 0 \tag{2.7}$$

where the notation $\big|_{(2.4)}$ means that the relation holds on any solution of Eq. (2.4). If one of the variables, for example x^1, is time t then the component C^1 is called the density of the conservation law.

Remark 2.1. In practical calculations the conservation law (2.7) can be rewritten to an equivalent form. If

$$C^1\big|_{(2.4)} = \widetilde{C}^1 + D_2(h^2) + \cdots + D_n(h^n).$$

then one obtains the following conservation law:

$$D_t(\widetilde{C}^1) + D_2(\widetilde{C}^2) + \ldots + D_n(\widetilde{C}^n) = 0$$

where

$$\widetilde{C}^2 = C^2 + D_t(h^2), \quad \ldots, \quad \widetilde{C}^n = C^n + D_t(h^n)$$

because $D_t D_i(h^i) = D_i D_t(h^i)$.

I have used this in my calculations of conservation laws.

By employing differential variables one can also rewrite Eq. (2.2) in the following form:

$$. \tag{2.8}$$

Thus, conserved $\left.\frac{dC}{dt}\right|_{(2.1)} \equiv D_t(C)\big|_{(2.1)} = 0$ quantities and conserved vectors can be computed with the help of Eq. (2.8) and Eq. (2.7), respectively (see, e.g. [27]–[31]).

2.2 Hamilton's principle and the Euler-Lagrange equations

Consider again a motion of a dynamical system with a kinetic energy $T(t,q,\dot{q})$ and a potential energy $U(t,q)$. The function

$$L(t,q,v) = T(t,q,\dot{q}) - U(t,q)$$

is called the Lagrangian of the system.

Hamilton's principle, or the principle of least action, states that the true motion of the system between two chosen times t_1 and t_2 is described by the fact that the trajectories of the particles provide an extremum of the action functional

$$\int_{t_1}^{t_2} \mathcal{L}(t,q,v)\,dt. \tag{2.9}$$

This requirement is equivalent to the statement that the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q^\alpha} - D_t\left(\frac{\partial \mathcal{L}}{\partial v^\alpha}\right) = 0, \quad \alpha = 1,\dots,s \tag{2.10}$$

hold. They give a necessary condition for $g(t)$ to provide an extremum of the integral (2.9).

In the case of several independent variables $x = (x^1,\dots,x^n)$ and dependent variables $u = (u^1,\dots,u^m)$ an action integral has the form

$$\int_V \mathcal{L}(x,u,u_{(1)},\dots,u_{(p)})\,dx \tag{2.11}$$

where V is an arbitrary n-dimensional volume in the space of the variables x and the Lagrangian L is a function depending on a finite number of differential variables. The corresponding Euler-Lagrange equations have the form:

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1,\dots,m, \tag{2.12}$$

where

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha} + \ldots + (-1)^s D_{i_1} D_{i_2} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}} + \ldots \quad (2.13)$$

is the variational derivative.

In my first two articles I discuss conservation laws for the Euler-Lagrange equations.

Definition 2.3. A conservation law is called a trivial conservation law if

$$D_i(C^i) \equiv 0$$

or C^i are smooth functions of $\frac{\delta L}{\delta u^\alpha}, \; D_i \frac{\delta L}{\delta u^\alpha}, \; \cdots$ Two conservation laws which only differ by a trivial conservation law are regarded as equivalent.

2.3 Lie group transformations and Noether's theorem

Assume that the Euler-Lagrange equations (2.12) admit a one-parameter Lie transformation group G, i.e. a local group of transformations

$$\bar{x} = \phi(x,u,a), \quad \bar{u} = \psi(x,u,a),$$

where

$$\phi = (\phi^1,\ldots,\phi^n), \quad \psi = (\psi^1,\ldots,\psi^m),$$

and

$$\phi(x,u,0) = x, \quad \psi(x,u,a) = u.$$

The infinitesimal generator of the group G has the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \quad (2.14)$$

where

$$\xi^i(x,u) = \frac{\partial \varphi^i(x,u,a)}{\partial a}\Big|_{a=0}, \quad \eta^\alpha(x,u) = \frac{\partial \psi^\alpha(x,u,a)}{\partial a}\Big|_{a=0}.$$

Definition 2.4. A variational integral (2.10) is invariant under the group G if

$$\int_{\bar{V}} \mathcal{L}(\bar{x},\bar{u},\bar{u}_{(1)},\ldots,\bar{u}_{(p)})\, d\bar{x} = \int_V \mathcal{L}(x,u,u_{(1)},\ldots,u_{(p)})\, dx.$$

The invariance condition is given by the following lemma.

Lemma 2.1. An integral (2.11) is invariant under the group G if and only if [15]

$$X(\mathrm{L}) + \mathrm{L}D_i(\xi^i) = 0. \quad (2.15)$$

Here X is a prolonged version of the generator (2.14):

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1 \ldots i_s}^\alpha \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha} + \cdots, \qquad (2.16)$$

where

$$\zeta_i^\alpha = D_i(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{ji}^\alpha,$$

$$\zeta_{i_1 \ldots i_s}^\alpha = D_{i_1} \ldots D_{i_s}(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{j i_1 \ldots i_s}^\alpha.$$

Noether proved her theorem by the application of the variational procedure to the integral of action. Using her idea Hill presented the explicit form of conserved quantities in the case of the first-order Lagrangians $L(x,u,u_{(1)})$ (see [21], Eq. (43)). In my articles I have used the following generalized form of Noether's theorem proved by Ibragimov [22], [32] on the basis of the group-theoretical approach.

Theorem 2.1. Let the variational integral (2.11) be invariant with respect to a group G with generators (2.14). Then a vector C with components

$$C^i = N^i(L), \qquad i = 1,2,\ldots,n, \qquad (2.17)$$

is a conserved vector for the Euler-Lagrange equations (2.12), i.e.

$$D_i(C^i)\big|_{2.12)} = 0. \qquad (2.18)$$

Here N^i are Ibragimov's operators [32], [15]:

$$N^i = \xi^i + W^\alpha \Big\{ \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{i j_1 \ldots j_s}^\alpha} \Big\}$$

$$+ \sum_{r \geq 1} D_{k_1} \ldots D_{k_r}(W^\alpha) \Big\{ \frac{\partial}{\partial u_{i k_1 \ldots k_r}^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{i k_1 \ldots k_r j_1 \ldots j_s}^\alpha} \Big\}, \qquad (2.19)$$

where $W^\alpha = \eta^\alpha - \xi^j u^\alpha_j$.

Corollary. If for some one-parameter transformation group the invariance condition (2.15) is not satisfied but the "divergence" condition

$$X(L) + L D_i(\xi^i) = D_i(B^i) \qquad (2.20)$$

holds, then the components of the corresponding conserved vector have the form:

$$C^i = N^i(L) - B^i, \qquad i = 1,2,\ldots,n. \qquad (2.21)$$

3 A basis of conservation laws

Besides the operator X, we shall also use its equivalent canonical LieBäcklund operator [33], [34]:

$$\overline{X} = X - \xi^j D_j = \bar{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + \bar{\zeta}_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \cdots + \bar{\zeta}_{i_1 \ldots i_s}^\alpha \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha} + \cdots , \quad (3.1)$$

where

$$\bar{\eta}^\alpha = W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \qquad \bar{\zeta}_i^\alpha = D_i(\bar{\eta}^\alpha), \qquad \bar{\zeta}_{i_1 \ldots i_s}^\alpha = D_{i_1} \ldots D_{i_s}(\bar{\eta}^\alpha).$$

Some conservation laws can be obtained more readily by using the commutativity of the generators $\overline{X}$ (or generators (2.14) with ξ^1 = const., ..., ξ^n = const.) and D_i.

Lemma 3.1. A canonical Lie–Bäcklund operator $\overline{X}$ and an operator of total differentiation D_i are commutative [15]:

$$\overline{X} D_i = D_i \overline{X}.$$

Lemma 3.2. If $C = (C^1, \ldots, C^n)$ satisfies a conservation law for some differential equation and a generator X is admitted by the equation in question then the vector with the components

$$C'^i = \overline{X}(C^i) \quad (3.2)$$

also satisfies a conservation law [15].

Hence lemmas 3.1 and 3.2 furnish the basis for another idea for calculating conservation laws. Moreover, conserved vectors can be computed for a differential equation without any Lagrangian if it has a known conservation law (see, e.g. [35] and [36]).

The property (3.2) makes it possible to introduce the concept of a basis (with respect to the group G) of the conservation laws and thus reduce the number of vectors C that must be constructed by means of Noether's theorem.

Definition 3.1. Let $\{C\}$ be a set of vectors satisfying the conservation law (2.18). A basis of the set $\{C\}$ is its minimal subset from which $\{C\}$ can be obtained by repeated application of (3.2) and by linear combinations. The conservation laws corresponding to the basis vectors form the basis of the conservation laws.

Using an example of gasdynamics equations, it was conjectured in [32] that the following diagram is commutative:

$$\begin{array}{ccc} X_1 & \xrightarrow{\mathrm{ad}X} & X_2 \\ N_1^i \big\downarrow & & \big\downarrow N_2^i \\ C_1 & \xrightarrow{\overline{X}} & C_2 \end{array}$$

and this statement can be used for construction of a basis of conserved vectors. The operators N_1^i and N_2^i in the diagram are given by (2.19) and X, X_1, X_2 by (2.14), the action ad$X$ is defined as follows:

$$\mathrm{ad}X(X_1) \equiv [X,X_1] = XX_1 - X_1X.$$

Hence the commutator $[X,X_1]$ has the form

$$[X, X_1] = \Big(X(\xi_1^i) - X_1(\xi^i)\Big)\frac{\partial}{\partial x^i} + \Big(X(\eta_1^\alpha) - X_1(\eta^\alpha)\Big)\frac{\partial}{\partial u^\alpha}\,. \tag{3.3}$$

Following this idea I verified the validity of the statement by means of several examples [37], [38] and then proved the following general result [39].

Theorem 3.1. Let generators X, X_1, X_2 of the form (2.14) be admitted by the Euler-Lagrange equations (2.12). Let the conserved vectors C_1, C_2 correspond (by Noether's theorem) to the generators X_1, X_2 and let

$$[X,X_1] = X_2.$$

Then the vectors $\overline{X}(C_1)$ and C_2 define equivalent conserved vectors, i.e.

$$\overline{X}(C_1) = C_2.$$

Remark 3.1. The theorem also holds when instead of the invariance condition (2.15) of a variational integral we have the "divergence" condition
(2.20.)

The proof of the theorem is given in [39]. Later this theorem was formulated in another form in [12]. Specific examples given in [37] were used by Tsujishita [40] as applications in modern formal differential geometry.

Let us consider several examples.

3.1 Classical mechanics

Let Eq. (2.3), $m\ddot{\mathrm{x}} = 0$, $\mathrm{x} = (x^1,x^2,x^3)$,

describe the free motion of a particle of mass m. The equation has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\, m|\dot{\mathbf{x}}|^2$$

and admits a 10-parameter point transformation group containing space translations with the generators

$$X_\mu = \frac{\partial}{\partial x^\mu}, \quad \mu = 1, 2, 3, \tag{3.4}$$

translation of time with the generator

$$X_4 = \frac{\partial}{\partial t}, \tag{3.5}$$

rotations of the vector x with the generators

$$X_{\mu\nu} = x^\nu \frac{\partial}{\partial x^\mu} - x^\mu \frac{\partial}{\partial x^\nu}, \quad \mu, \nu = 1, 2, 3, \tag{3.6}$$

and Galilean transformations with the generators

$$X_{\mu 4} = t \frac{\partial}{\partial x^\mu}, \quad (\mu = 1, 2, 3). \tag{3.7}$$

Hence according to Noether's theorem Eq. (2.3) has 10 conservation laws of the form

$$D_t(C)\big|_{(2.3)} = 0 \tag{3.8}$$

defined by the following conserved quantities: the linear momentum

$$\mathbf{p} = m\dot{\mathbf{x}},$$

the energy

$$E = \frac{1}{2} m|\dot{\mathbf{x}}|^2$$

the angular momentum

$$\mathbf{M} = \mathbf{p} \times \mathbf{x}$$

and the vector $\mathbf{q} = m(\mathbf{x} - \dot{\mathbf{x}}t)$.

From the table of commutators, Table 3.1, it is easy to notice that only X_4 and one of generators $X_{\mu\nu}$ can not be obtained by using ad$X$. Thus, employing Theorem 3.1 we can conclude the following:

A basis of conservation laws consists of two conservation laws defined by the energy E and one of the components of the angular momentum $\mathbf{M}$.

Indeed, if we choose as a basis of conserved quantities E and, e.g. M^1 we can obtain other conserved quantities by means of the generators $X_{\mu 4}$ and $X_{\mu\nu}$ written in the prolonged form:

$$X_{\mu 4} = t \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial \dot{x}^\mu} \quad (\mu = 1, 2, 3)$$

and

$$X_{\mu\nu} = x^{\nu}\frac{\partial}{\partial x^{\mu}} - x^{\mu}\frac{\partial}{\partial x^{\nu}} + \dot{x}^{\nu}\frac{\partial}{\partial \dot{x}^{\mu}} - \dot{x}^{\mu}\frac{\partial}{\partial \dot{x}^{\nu}} \quad (\mu,\nu = 1,2,3). \tag{3.9}$$

Then

$$X_{\mu 4}E = p_{\mu}, \qquad X_{24}M_1 = q_3,$$

$$X_{12}(M_1) = M_2, \qquad X_{13}(M_1) = M_3.$$

$$X_{34}M_2 = q_1, \qquad X_{14}M_3 = q_2.$$

Table 3.1: Table of commutators (Classical mechanics)

	X_1	X_2	X_3	X_4	X_{12}	X_{23}	X_{13}	X_{14}	X_{24}	X_{34}
X_1	0	0	0	0	$-X_2$	0	$-X_3$	0	0	0
X_2		0	0	0	X_1	$-X_3$	0	0	0	0
X_3			0	0	0	X_2	X_1	0	0	0
X_4				0	0	0	0	X_1	X_2	X_3
X_{12}					0	$-X_{13}$	X_{23}	X_{24}	$-X_{14}$	0
X_{23}						0	$-X_{12}$	0	X_{34}	$-X_{24}$
X_{13}							0	X_{34}	0	$-X_{14}$
X_{14}								0	0	0
X_{24}									0	0
X_{34}										0

3.2 Relativistic mechanics

The equation of free motion of a relativistic particle in the Minkowski space with the metric

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

has the Lagrangian

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}, \quad v^i = \frac{dx^i}{dt} \quad (i = 1,2,3),$$

where c is a constant equal to the light velocity in vacuum,

$$x^1 = x,\ x^2 = y,\ x^3 = z.$$

It admits the 10-parameter non-homogeneous Lorentz group with the generators (3.4)–(3.6) and the generators of the Lorentz transformations

$$X_{\mu 4} = x^4 \frac{\partial}{\partial x^\mu} + \frac{1}{c^2} x^\mu \frac{\partial}{\partial x^4}, \quad (\mu = 1, 2, 3) \tag{3.10}$$

where $x^4 = t$.

The corresponding conservations laws have the form similar to (3.8). According to Noether's theorem they are defined by the following conserved quantities:

$$\mathbf{p}_0 = mc\dot{\mathbf{x}}, \qquad E_0 = mc^3\dot{x}^4, \qquad \mathbf{M}_0 = \mathbf{p}_0 \times \mathbf{x}, \qquad \mathbf{Q}_0 = mc(\mathbf{x}\dot{x}^4 - \dot{\mathbf{x}}x^4) \tag{3.11}$$

Table 3.2: Table of commutators (Relativistic mechanics)

	X_1	X_2	X_3	X_4	X_{12}	X_{23}	X_{13}	X_{14}	X_{24}	X_{34}
X_1	0	0	0	0	$-X_2$	0	$-X_3$	$\frac{1}{c^2}X_4$	0	0
X_2		0	0	0	X_1	$-X_3$	0	0	$\frac{1}{c^2}X_4$	0
X_3			0	0	0	X_2	X_1	0	0	$\frac{1}{c^2}X_4$
X_4				0	0	0	0	X_1	X_2	X_3
X_{12}					0	$-X_{13}$	X_{23}	X_{24}	$-X_{14}$	0
X_{23}						0	$-X_{12}$	0	X_{34}	$-X_{24}$
X_{13}							0	X_{34}	0	$-X_{14}$
X_{14}								0	$-\frac{1}{c^2}X_{12}$	$-\frac{1}{c^2}X_{13}$
X_{24}									0	$-\frac{1}{c^2}X_{23}$
X_{34}										0

where $\mathbf{x} = (x^1, x^2, x^3)$ and the dot denotes differentiation with respect to the length of the arc s in the Minkowski space. Comparing Table 3.2 and Table 3.1, one can see a significant difference. Namely, in the case of relativistic mechanics the time translation generator X_4 can be obtained from other operators by using ad$X$. Therefore, employing Theorem 3.1 we can conclude that

A basis of conserved quantities (3.11) (with respect to group G) is defined by one conserved quantity, e.g. any of the components of the angular momentum $\mathbf{M}_0$.

Indeed, if we choose M_0^1 as a basis of conserved quantities, under the action of the generators of the Lorentz transformations written in the prolonged form

$$X_{\mu 4} = x^4 \frac{\partial}{\partial x^\mu} + \frac{1}{c^2} x^\mu \frac{\partial}{\partial x^4} + \dot{x}_4 \frac{\partial}{\partial \dot{x}^\mu} + \frac{1}{c^2} \dot{x}^\mu \frac{\partial}{\partial \dot{x}^4}, \quad (\mu = 1, 2, 3) \tag{3.12}$$

and the generators of rotation (3.9) we obtain:

$$X_{24}(M_0^1) = Q_0^3, \quad X_{12}(M_0^1) = M_0^2, \quad X_{13}(M_0^1) = M_0^3.$$

Then we have

$$X_{34}(M_0^2) = Q_0^1, \quad X_{14}(M_0^3) = Q_0^2,$$

E_0 can be obtained from Q_0 with the help of the translation generators X_μ, i.e.

$$c^2 X_\mu(Q_0^\mu) = E_0$$

and the energy E_0 transforms into the momentum $\mathbf{p}_0$,

$$X_{\mu 4}(E_0) = \mathbf{p}_0^\mu.$$

Remark 3.2. On the other hand it is also possible to choose any component of the vector Q_0 as a basis of conserved quantities.

3.3 Motion in the de Sitter space

Consider the space V_4 with the metric

$$ds^2 = \frac{1}{\Phi^2}(c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{3.13}$$

where

$$\Phi = 1 + \frac{K}{4}r^2, \quad r^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{3.14}$$

and K = const. denotes the curvature of the Sitter space-time.

As well as the equation of free motion of a particle in Minkowski space a similar equation in the de Sitter space has the Lagrangian

$$\mathcal{L} = -\frac{mc^2}{\Phi}\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}, \quad v^\mu = \frac{dx^i}{dt} \quad (\mu = 1, 2, 3),$$

also admits 10-parameter group G with the generators of rotations and the generators of Lorentz transformations of the form

$$X_{\mu\nu} = x^\nu \frac{\partial}{\partial x^\mu} - x^\mu \frac{\partial}{\partial x^\nu} \quad (\mu < \nu, \quad \mu = 1, 2, 3; \ \nu = 1, 2, 3, 4), \tag{3.15}$$

but the generators of space translations (3.4) and translations of time (3.5) are replaced by the generators

$$X_\nu = \left[\frac{K}{2}x^\nu x^i + (\Phi - 2)\delta^{\nu i}\right]\frac{\partial}{\partial x^i}, \quad \nu, i = 1, 2, 3, 4. \tag{3.16}$$

Here the generators are written in the coordinates

$$x^1 = x, \qquad x^2 = y, \qquad x^3 = z, \qquad x^4 = ict$$

and δ^{ki} is a Kronecker symbol.

According to Noether's theorem there are 10 conserved quantities similar to (3.11), the linear momentum p_k, the energy E_K, the angular momentum M_K and the vector Q_K.

The structure of Lie algebra with the basis (3.15)–(3.16) is determined by the commutators

$$[X_\mu, X_\nu] = KX_{\mu\nu}, \quad [X_\mu, X_{\mu\nu}] = X_\nu, \quad [X_{\mu\nu}, X_\alpha] = 0 \quad (\alpha \neq \mu, \alpha \neq \nu)$$

$$[X_{\mu\nu}, X_{\alpha\beta}] = \delta_{\mu\alpha}X_{\nu\beta} + \delta_{\nu\beta}X_{\mu\alpha} - \delta_{\mu\beta}X_{\nu\alpha} - \delta_{\nu\alpha}X_{\mu\beta}. \quad ,$$

Hence, for the equation of free motion of a particle in the de Sitter space, we arrive at the following assertion:

A basis of conserved quantities with respect to the group G is defined by one conserved quantity, e.g. any of the components of the angular momentum $M_K = M_0/\Phi^2$.

Remark 3.3. In this case any conserved quantity, i.e the energy, or any component of the linear momentum or any component of the vector Q_K can also be chosen as a basis of the conserved quantities.

3.4 Nonlinear wave equation

The equation

$$u_{tt} - \Delta u + \lambda u^3 = 0 \qquad (3.17)$$

has the Lagrangian

$$\mathcal{L} = |\Delta u|^2 - u_t^2 + \frac{1}{2}\lambda u^4$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$, $\lambda =$ const.. Eq. (3.17) describes string vibration immersed in nonlinear medium. Eq. (3.17) is also used in quantum nonlinear field theory. It admits the 15-dimensional group of conformal transformations in the Minkowski space and has, correspondingly, 15 conservation laws of the form

$$D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = 0.$$

The basis of conserved vectors for the nonlinear wave equation also consists of one conserved vector [37], [38].

3.5 Lin–Reissner–Tsien equation

The equation [41]

$$-\phi_x\phi_{xx} - 2\phi_{xt} + \phi_{yy} = 0$$

describes the non-steady-state potential gas flow with transonic velocities. It has the Lagrangian

$$\mathcal{L} = |\Delta u|^2 - u_t^2 + \frac{1}{2}\lambda u^4, \quad \Delta u = u_{xx} + u_{yy}, \quad \lambda = \text{const.}$$

and admits an infinite transformation group [42]. Accordingly, by Noether's theorem, the family of conservation laws [43] is infinite. Meanwhile, employing Theorem 3.1 we can conclude the following:

The basis of conservation laws consists of one conservation law [37], namely

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0$$

where

$$C^1 = \frac{1}{2}\varphi_y^2 - \frac{1}{6}\varphi_x^3, \quad C^2 = \varphi_t^2 + \frac{1}{2}\varphi_x^2\varphi_t, \quad C^3 = -\varphi_t\varphi_y.$$

3.6 Transonic three-dimensional gas motion

The equation

$$-u_x u_{xx} - 2u_{xt} + u_{yy} + u_{zz} = 0$$

of transonic gas motion has the Lagrangian

$$\mathcal{L} = \frac{1}{6}u_x^3 + u_x u_t - \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2$$

and admits an infinite group transformations (the generators [44], [45], [39] depends on arbitrary functions).

The basis of conservation laws

$$D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = 0$$

is defined by two vectors A_1 and A_4 where

$$A_1^1 = \frac{1}{6}u_x^3 - \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2, \quad A_1^2 = -u_t\left(\frac{1}{2}u_x^2 + u_t\right), \quad A_1^3 = u_t u_y, \quad A_1^4 = u_t u_z$$

$$A_4^1 = \bar{\eta}u_x, \quad A_4^2 = \bar{\eta}\left(\frac{1}{2}u_x^2 + u_t\right), \quad A_4^3 = -\bar{\eta}u_y + z\mathcal{L}, \quad A_4^4 = -\bar{\eta}u_z - y\mathcal{L}$$

$$\bar{\eta} = yu_z - zu_y.$$

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3.7 Short waves

During first underwater nuclear and thermonuclear explosions near the arctic island Novaja Zemlja in the USSR it was discovered that weak waves were drastically increasing the destructive force of a shock wave [46]. Rizhov and Khristianovich [47] presented the equations describing the behavior of these so-called "short waves".

The equations of short waves

$$u_y - 2v_t - 2(v - x)v_x - 2kv = 0, \qquad v_y + u_x = 0, \qquad k = \text{const.},$$

admit an infinite-dimensional group [48]. They can be reduced by the substitution $u = \phi_y$, $v = -\phi_x$ to the equation

$$\phi_{yy} + 2\phi_{xt} - 2(x + \phi_x)\phi_{xx} + 2k\phi_x = 0, \tag{3.18}$$

which has the Lagrangian

$$\mathcal{L} = (\varphi_t\varphi_x - \frac{1}{3}\varphi_x^3 - x\varphi_x^2 + \frac{1}{2}\varphi_y^2)\exp[2(k+1)t].$$

In [39] I calculated the following generators for Eq. (3.18):

$$X_1 = 9\frac{\partial}{\partial t} - 4(k+1)x\frac{\partial}{\partial x} - 2(k+1)y\frac{\partial}{\partial y} - 8(k+1)\varphi\frac{\partial}{\partial \varphi}$$

$$X_2 = -\mu' y\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} - \left[xy(\mu'' + \mu') - \frac{y^3}{3}\, d_k\mu'\right]\frac{\partial}{\partial \varphi}, \quad , \tag{3.19}$$

(3.20)

$$X_3 = \kappa\frac{\partial}{\partial x} + \left[y^2 d_k\kappa - x(\kappa' + \kappa)\right]\frac{\partial}{\partial \varphi}, \tag{3.21}$$

$$X_4 = \lambda y\frac{\partial}{\partial \varphi}, \quad X_5 = \sigma\frac{\partial}{\partial \varphi}, \quad X_0 = y\frac{\partial}{\partial t} \tag{3.22}$$

where $\mu,\kappa,\lambda,\sigma$ are arbitrary functions of t, the prime denotes differentiation with respect to t, $d_k = d^2/dt^2 + (k+1)d/dt + k$.

Although the constant k in Eq. (3.18) takes only the values 0 and 1 in accordance with the physical content of the problem, it can be regarded as an arbitrary parameter. For $k = 2;1/2$ there is an extension of the group, the following generators are added:

$$X_6 = 9a\frac{\partial}{\partial t} + \{[3a' - 4(k+1)a]x - [3a'' - (k+1)a']y^2\}\frac{\partial}{\partial x} + [6a' - 2(k+1)a]y\frac{\partial}{\partial y}$$
$$+\{-[3a' + 8(k+1)a]\varphi - \frac{x^2}{2}[3a'' + (5-4k)a'] + [3a''' + (2-k)a'' - (k+1)a']xy^2$$
$$-\frac{y^4}{6}[3a^{\mathrm{IV}} + 2(k+1)a''' - (k^2 - k + 1)a'' - k(k+1)a']\}\frac{\partial}{\partial \varphi}.$$

The operator X_0 does not satisfy the conditions of Noether's theorem and the generators X_4 and X_5 give only trivial conservation laws.

Among the commutation relations for X_1, X_2, X_3, X_6 we can distinguish

$$\text{ad}\, X_2(X_1) = -X_2 < 9\mu' + 2(k+1)\mu >,$$

$$\text{ad}\, X_3(X_1) = -X_3 < 9\kappa' + 4(k+1)\kappa >, \quad \text{ad}\, X_6(X_1) = -X_6 < 9a' > .$$

Here the brackets < ... > mean that instead of the arbitrary function occurring in the coordinates of the generator (or conserved vector) it is necessary to substitute the expression in these brackets.

Therefore, the basis of the conservation laws

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0$$

is determined by one vector corresponding to X_1, i.e. A_1 with coordinates

$$A^1{}_1 = \bar{\eta}_1 E\phi_x + 9L, \qquad A^2{}_1 = \bar{\eta}_1 E(\phi_t - \phi_x^2 - 2x\phi_x) - 4(k+1)xL,$$

$$A^3{}_1 = \bar{\eta}_1 E\phi_y - 2(k+1)yL, \qquad \bar{\eta}_1 = -8(k+1)\phi - 9\phi_t + 4(k+1)x\phi_x + 2(k+1)y\phi_y.$$

3.8 Dirac equations

The Dirac equations

$$\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi = 0, \qquad \frac{\partial \widetilde{\psi}}{\partial x^k}\gamma^k - m\widetilde{\psi} = 0 \quad , \tag{3.23}$$

are the relativistic quantum mechanical wave equations used for the description of fermions, i.e. elementary particles having half-integer spin number (say, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$). Eqs (3.23) have the Lagrangian

$$\mathcal{L} = \frac{1}{2}\Big\{\widetilde{\psi}\Big(\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi\Big) - \Big(\frac{\partial \widetilde{\psi}}{\partial x^k}\gamma^k - m\widetilde{\psi}\Big)\psi\Big\}, \quad k = 1, 2, 3, 4 .$$

Here the independent variables are

$$x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = ict,$$

the dependent variables

$$\psi = (\psi^1, ..., \psi^4) \quad \widetilde{\psi} = (\widetilde{\psi}^1, ..., \widetilde{\psi}^4)$$

are 4-dimensional complex vectors and γ^k are 4×4 and complex matrices.

ENTRI

The maximal group admitted by Eqs. (3.23) is obtained in [22]. The Dirac equations have an infinite number of conservation laws

$$D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = 0.$$

Using Theorem 3.1 we obtain the following result [39].

For $m = 0$ the basis of conserved vectors is formed by 3 vectors, A_{12}, A_5, A_8, and for $m \neq 0$ by 2 vectors A_{12}, A_5.
Their coordinates have the form

$$A_{12}^k = \frac{1}{4}\{\widetilde{\psi}(\gamma^k\gamma^1\gamma^2 + \gamma^1\gamma^2\gamma^k)\psi\} + x^2A_1^k - x^1A_2^k$$

$$A_5^k = -i\widetilde{\psi}\gamma^k\psi, \qquad A_8^k = i\widetilde{\psi}\gamma^k\gamma^5\psi,$$,

where

$$A_l^k = \frac{1}{2}\left\{\frac{\partial\widetilde{\psi}}{\partial x^l}\gamma^k\psi - \widetilde{\psi}\gamma^k\frac{\partial\psi}{\partial x^l}\right\} + \delta_l^k, \quad k = 1,2,3,4, \quad l = 1,2$$,

and δ_l^k is a Kronecker's symbol.

4 Equations without Lagrangians

4.1 Formal Lagrangian

Many differential equations cannot be formulated as the Euler–Lagrange equations since they have no Lagrangians. Therefore, it is impossible to apply Noether's theorem for calculating conservation laws. However, according to [49] and [50], it is possible to introduce a formal Lagrangian if any given system of equations is taken into consideration together with the adjoint system. In his recent paper [50] Ibragimov has proved that the adjoint system inherits symmetries of the given system and has suggested a new theorem on nonlocal conservation laws.

Consider an arbitrary system of *s*th-order partial differential equations

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m. \tag{4.1}$$

where the functions $F_\alpha(x,u,u_{(1)},\ldots,u_{(s)})$ depend on n independent variables $x = (x^1,\ldots,x^n)$, m dependent variables $u = (u^1,\ldots,u^m)$, $u = u(x)$, and their derivatives up to an arbitrary order s.

Definition 4.1. The adjoint system to Eqs (4.1) is defined by [51]

$$F^*_\alpha\big(x,u,v,u_{(1)},v_{(1)},\dots,u_{(s)},v_{(s)}\big) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1,\dots,m, \quad (4.2)$$

where $v = (v^1,\dots,v^m)$ are new dependent variables, $v = v(x)$, and $\frac{\delta}{\delta u^\alpha}$ is the variational derivative (2.13).

In the case of linear equations this definition is equivalent to the standard one.

Remark 4.1. The variables $v = (v^1,\dots,v^m)$ were called in [50] nonlocal variables in accordance with the general concept of nonlocal symmetries. Therefore, conservation laws involving v were named nonlocal conservation laws.

Using the new definition of the adjoint system, it can be shown that any system of sth-order differential equations (4.1) considered together with its adjoint equation (4.2) has a Lagrangian. Namely, the Euler-Lagrange equations with the Lagrangian

$$\mathcal{L} = v^\beta F_\beta\big(x,u,u_{(1)},\dots,u_{(s)}\big) \quad (4.3)$$

provide the simultaneous system of equations (4.1), (4.2) with $2m$ dependent variables $u = (u^1,\dots,u^m)$ and $v = (v^1,\dots,v^m)$.

Definition 4.2. The system (4.1) is called self-adjoint if the substitution $v = u$ gives

$$F^* = \lambda(x,u,u_{(1)},\dots,u_{(s)})F. \quad (4.4)$$

The system (4.1) is called quasi-self-adjoint [52] if there exists a function $h(u)$ such that Eq (4.4) holds upon the substitution $v = h(u)$.

4.2 Maxwell-Dirac equations

We have the system of equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} + \sigma_m \mathbf{B} = 0,$$

$$\nabla \times \mathrm{B} - \frac{\partial \mathrm{E}}{\partial t} - \sigma_e \mathrm{E} = 0, \quad (4.5)$$

$$\nabla \cdot \mathrm{E} - \rho_e = 0,$$

$$\nabla \cdot B - \rho_m = 0,$$

where σ_m, σ_e = const. The system (4.5) has eight equations for eight dependent variables: six coordinates of the electric and magnetic vector fields $E = (E^1, E^2, E^3)$ and $B = (B^1, B^2, B^3)$, respectively, and two scalar quantities ρ_e and ρ_m, the electric and magnetic monopole charge densities.

Using (4.3) we write the Lagrangian (4.3) for Eqs. (4.5) in the following form [54] :

$$\mathcal{L} = \mathbf{V} \cdot \Big(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} + \sigma_m \mathbf{B}\Big) + R_e\Big(\nabla \cdot \mathbf{E} - \rho_e\Big)$$

$$+\mathbf{W} \cdot \Big(\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} - \sigma_e \mathbf{E}\Big) + R_m\Big(\nabla \cdot \mathbf{B} - \rho_m\Big), \quad (4.6)$$

where V, W, R_e, R_m are adjoint variables. With this Lagrangian the adjoint equations for the new dependent variables V, W, R_e, R_m have the form [55]

$$\nabla \times \mathbf{V} + \frac{\partial \mathbf{W}}{\partial t} - \sigma_e \mathbf{W} = 0,$$

$$\nabla \times \mathbf{W} - \frac{\partial \mathbf{V}}{\partial t} + \sigma_m \mathbf{V} = 0, \quad (4.7)$$

$$R_e = 0, \qquad R_m = 0.$$

4.3 Conservation laws

Each generator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

admitted by a first-order system

$$F_\alpha(x,u,u_{(1)}) = 0, \qquad \alpha = 1,\dots,m$$

leads to a conserved vector with the components

$$C^i = v^\beta\Big[\xi^i F_\beta + (\eta^\alpha - \xi^j u_j^\alpha \frac{\partial F_\beta}{\partial u_i^\alpha})\Big] \quad (4.8)$$

where $i = 1,\dots,n$ and v^β solve the adjoint system

$$F_\alpha^*(x,u,u_{(1)},v_{(1)}) = 0, \qquad \alpha = 1,\dots,m.$$

The conservation law for Eqs (4.5) has the form

$$D_t(\tau) + \mathrm{div}\chi = 0, \tag{4.9}$$

which holds on the solutions of Eqs (4.5) and (4.7). Here τ is the density of the conservation law (4.9), $\chi = (\chi^1, \chi^2, \chi^3)$, and

$$\mathrm{div}\chi \equiv \nabla \cdot \chi = D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3).$$

The Maxwell-Dirac equations are neither self-adjoint nor quasi-self-adjoint. Consequently, the conservation laws obtained by using Eqs (4.8) are nonlocal.

4.4 General magma equation

The equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial z}\left\{f^n\left[1 - \frac{\partial}{\partial z}\left(\frac{1}{f^m}\frac{\partial f}{\partial t}\right)\right]\right\} = 0 \tag{4.10}$$

models the migration of melt through the Earth's mantle. It follows from the equations

$$u_z = -f_t, \quad u = f^n\left[1 + \frac{\partial}{\partial z}\left(\frac{1}{f^m}\frac{\partial u}{\partial z}\right)\right] \tag{4.11}$$

where u is the vertical barometric flux of melt, f is the volume fraction of melt, z is a vertical space coordinate and t is time. All the variables are dimensionless. Eqs. (4.11) were proposed by Scott and Stevenson [56]. They suggested that $2 \le n \le 5$ or even bigger and supposed that $0 \le m \le 1$.

Some authors discussed Eq. (4.10) for any values of n and m.

I denote f by u in order to make Eq. (4.10) compatible with the general notation used above. It has the form

$$F \equiv u_t + D_z\left\{u^n\left[1 - D_z\left(u^{-m}u_t\right)\right].\right\} = 0 \tag{4.12}$$

The general magma equation does not have any Lagrangian and therefore the formal Lagrangian is introduced. Using the Lagrangian and employing infinitesimal symmetries of Eq. (4.10) nonlocal conservation laws are obtained in my articles [57]– [59]. The central part of these articles is the proof of the remarkable property of Eq. (4.12) to be quasi-self-adjoint for any values of the parameters m and n. This property allows us to obtain local conservation laws from nonlocal ones. They include the local conservation laws obtained by the direct method by Barcilon and Richter [27] and Harris [28] and later discussed in [36].