The semi-empirical mass formula

1 Introduction

The liquid drop model reproduces the gross features of the binding energy but does not give the dependences on the individual numbers of protons and neutrons. For this we need to add more terms which result in the "semi-empirical mass formula".

2 The semi-empirical mass formula

We saw the binding energy can be expressed as

$$B_E = a_v A - a_s A^{2/3}$$

where a_v (the volume term constant) and a_s (the surface term constant) are parameters to be determined from data.

   The above expression for the binding energy is a reasonable approximation but does not reproduce a lot of the effects we have already seen. It predicts that the ground state binding energy only depends on A but is independent of Z or N, which we have already seen is not correct. It ignors quantum effects such as the Pauli exclusion principle (and others) completely. To get a better agreement we need to add three more terms which explicitly depend on Z and N.

   The first term we will now add is easy to understand. We know there is an EM repulsive force between protons due to their charge and so this will reduce the binding energy for nucleons with several protons. As we believe the nuclear force itself is independent of nucleon type, then the protons will on average be spread evenly throughout the nucleus, which means the charge density is uniform. It is a standard problem in electrostatics to calculate the energy required to assemble a sphere of uniform charge density and the result is

$$\frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 r}$$

For a nucleus with Z protons, this self-energy is therefore

$$\Delta B_E = -\frac{3}{5}\frac{e^2}{4\pi\epsilon_0 r_0}\frac{Z^2}{A^{1/3}} = -a_c\frac{Z^2}{A^{1/3}}$$

where a_c = 0.72 MeV. Note the dependence on Z^2, not Z; the EM force is long range and so every proton affects every other proton in the nucleus, not just its nearest neighbours. Contrast this to the short range nuclear force where the nucleons only affect their nearest neighbours and the energy depends on A. In fact, although this Z^2 form is often used, strictly speaking it is not quite right. The energy given by the expression above is that needed to spread all the charge out throughout all space to an infinitely small density. However, the binding energy is defined as the energy need to break the nucleus into its constituent nucleons, i.e. break it into neutrons and protons, but *not* to spread the individual proton charges out. Indeed, the equation says even one proton, i.e. Z = 1, gives a correction to the binding energy, even though there is nothing to repel it. This means the correction to the binding energy should not be quite as big. A better form is $a_c Z(Z$

− 1)$/A^{1/3}$, which is now zero for $Z = 1$. Clearly, for large Z, as found in large nuclei, these two are very similar. We now have

$$B_E = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}}$$

This by itself breaks the independence on Z but clearly predicts the biggest binding energy for any A will be for $Z = 0$ or 1. We know this isn't right as we have stable nuclei with all values of charge up to $Z > 100$ in the atomic periodic table. We need to add two more terms to account for quantum effects.

The next term we will consider is called the "asymmetry term". The idea here is identical to the concept of a Fermi level in the physics of materials. The nucleons have energy levels in the nucleus and, being spin 1/2 particles, then each level can take two of each type of nucleon.
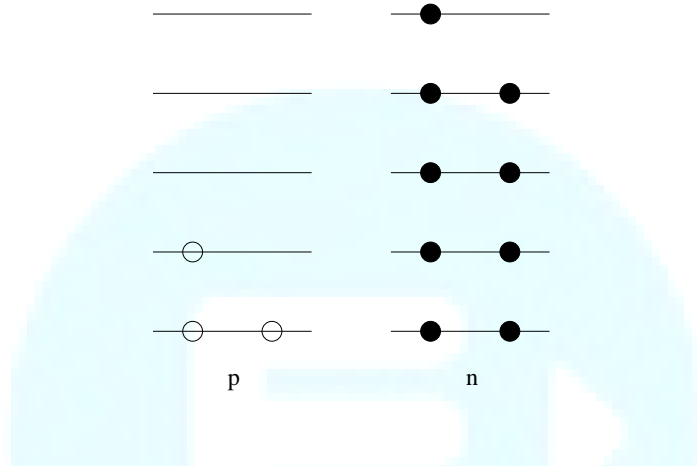
If we really tried to form a nucleus purely from neutrons, as implied by the terms we have so far for the binding energy, they would have to be put into higher and higher energy levels and so would be less and less strongly bound, reducing the binding energy. Clearly, putting protons into the nucleus instead would be beneficial for the binding energy as they could go into the deepest empty proton levels. It is clear the best situation is when the two are evenly balanced with $N = Z$. The details of the exact energy levels and numbers per level will be messy and vary with A, but a reasonable parametrisation turns out to given by $\Delta B_E \propto -(N-Z)^2$, i.e. the binding energy is reduced symmetrically for either $N > Z$ or $Z > N$. In fact, the spacing between states depends inversely on the size of the nucleus (as it does for a square well potential) so that larger nuclei have less of a binding energy loss if $N = 6$ Z, hence, the full term used is $-a_a(N-Z)^2/A$.

The final term which is needed is called the "pairing term". This occurs because of the different overlap of wavefunctions for pairs of nuclei in various states. For two identical nucleons in the same spatial state, with opposite spins to be antisymmetric as required, then the spatial wavefunctions are effectively identical and have maximal overlap. Because of the short range force, this gives more of a binding energy for this particular pair. This effect occurs for all nucleons except potentially the ones in the highest occupied energy level for each type of nucleon, where there is either one or two nucleons of that type. Hence, the nucleus will be more strongly bound for ones with an even number of nucleons of either type. There are three cases

1. Even-even, meaning an even number of both protons and neutrons, and hence even A. This has both pairs strongly bound.

2. Odd-odd, meaning an odd number of both protons and neutrons, and hence also even A. This is the least strongly bound.

3. Even-odd, meaning an even number of one type and an odd number of the other, and hence odd A. This has one strongly bound pair and so should be half way in between the previous two.

This is therefore simply parametrised by a form $a_p/A^{1/2}$, where a_p is takes its positive value for even-even nuclei, its negative value for odd-odd nuclei and is zero for even-odd nuclei. Note, the pairing term implies even-even nuclei always have the spins of the nucleons in the same spatial state parallel, so all such nuclei would be expected to have ground states with total spin zero; this is observed to be true.

Therefore, the total expression for the binding energy is

$$B_E = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + a_p \frac{1}{A^{1/2}}$$

and this is the semi-empirical mass formula. The best-fit parameters take values around $a_v = 15.8$ MeV, $a_s = 18.3$ MeV, $a_c = 0.71$ MeV, $a_a = 23.2$ MeV and $a_p = \pm 11.2$ MeV.

3 The beta-stability curve

The semi-empirical mass formula is a function of two variables, as $A = N + Z$. It gives the binding energy of the ground state of any nucleus, i.e. any values of Z and N. However, we have not observed nuclei with most of the combinations of Z and N which might be thought possible in the whole of the Z,N plane because the majority of them are highly unstable. The binding energy is largest in a specific region of the Z,N plane which is called the beta-stability valley ("valley" as the mass is at a minimum, "beta-stable" for reasons which will become clear in a later lecture).

It is easiest to analyse the binding energy along lines of constant A and we will look at it as a function of Z. Putting $N = A - Z$, then

$$B_E = \left(a_v A - a_s A^{2/3} - a_a A + \frac{a_p}{A^{1/2}} \right) + \left(\frac{a_c}{A^{1/3}} + 4a_a \right) Z + \left(-\frac{a_c}{A^{1/3}} - \frac{4a_a}{A} \right) Z^2$$

For odd A, when the pairing term a_p is zero, then this is a quadratic
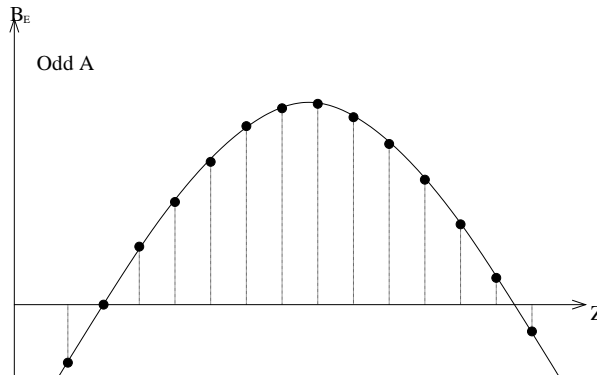
There is a particular value of Z for which the binding energy is maximum, but away from this value, the binding energy falls and eventually goes negative at which point the nucleus is no longer bound. Even within the region of positive binding energy, the non-maximum values of Z are not necessarily stable; as we will see, beta decay allows them to change protons to neutrons and vice versa, and so move along the curve to the maximum value. Hence, we only see the reasonably long-lived nuclei which are at or near the maximum.

For even A, then as Z changes by one, then Z (and N, given that A is fixed and even) goes from even to odd or vice versa. This means the pairing term changes sign. Hence, this shifts the quadratic curve up and down by $\pm a_p/A^{1/2}$ for the alternating even and odd Z values. However, the same stability arguments still hold; only nuclei near the peak live long enough to be seen.
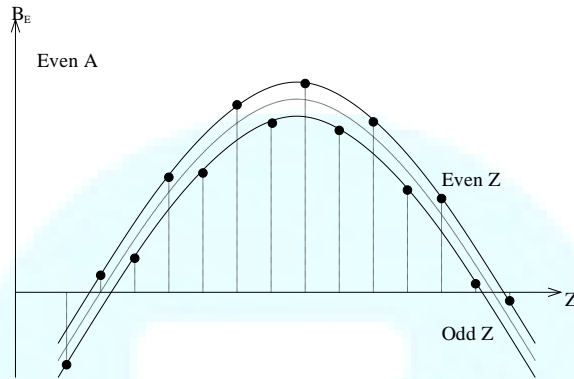
We need to know how the Z position of the peak changes with A. In principle we can approximate Z as continuous and take derivatives of the above quadratic function to find the maximum as a function of A. However, it is also useful to get an intuitive feel for what we would expect. The two terms driving this are the Coulomb and asymmetry terms; only a_c and a_a appear in the Z and Z^2 terms of the quadratic. If $a_c = 0$, then clearly $N = Z = A/2$ gives the maximum binding energy as the nuclei like to have equal numbers of levels filled. Conversely, if $a_a = 0$, then $Z = 0$ or 1 to minimise the Coloumb term and hence maximise the binding energy by limiting the amount of Coulomb repulsion. Hence, with both terms, we would generally expect that Z would be somewhat less than $A/2$ and N would be more than $A/2$. The relative size of these two terms is not the same for all A. The asymmetry term falls off as $1/A$ while the

Coulomb terms falls off only as $1/A^{1/3}$. Hence the latter becomes more important as A increases. Conversely, for small A, particularly as $a_c \approx 0.7$ MeV $\ll a_a \approx 23.2$ MeV, the Coulomb term has little effect. Therefore, we expect $Z \approx A/2$ for small A but $Z < A/2$ for large A. The actual stable and observed nuclei which form the beta-stability valley are shown below
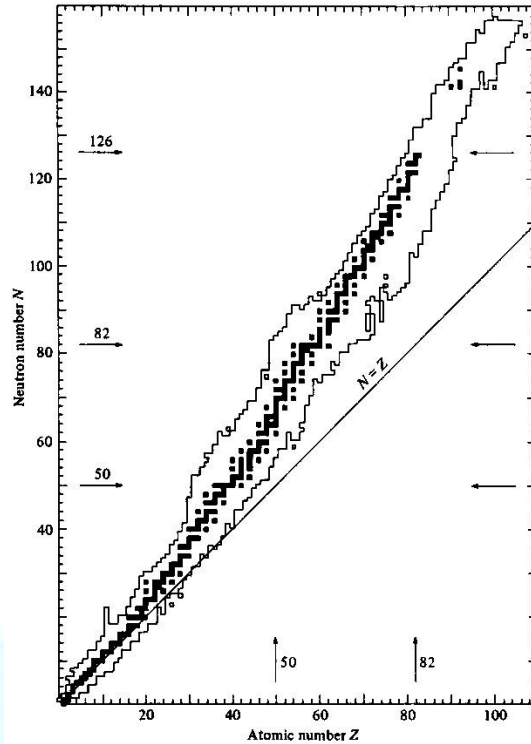
The line of the most stable nuclei is call the beta-stability curve. Note, there can be several stable nuclei for a given A. Nuclei lying on the beta-stability curve are often studied. For example, the binding energy per nucleon of these as a function of A are shown below and compared with the semi-empirical mass formula.

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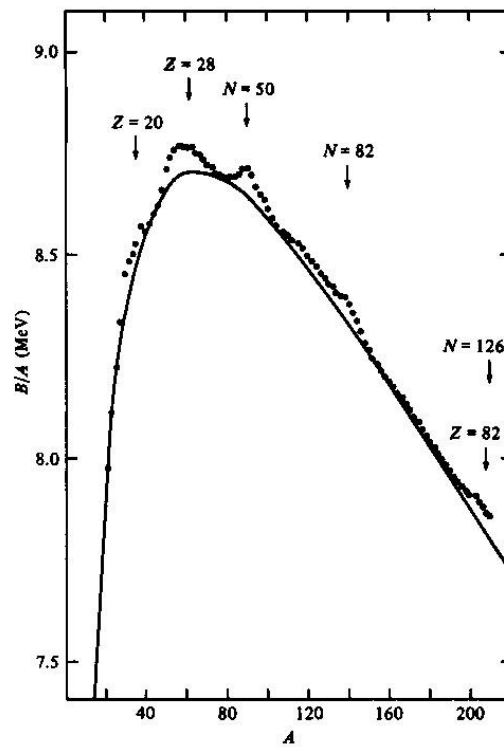
The agreement is generally very good. Note, the binding energy per nucleon is reasonably constant, between 7.8 and 8.8 MeV for all nuclei with $A > 30$. It peaks around $^{56}_{26}$Fe, which means this is the most strongly bound nucleus. It is therefore energetically favourable both to break up heavier nuclei to bring them closer to Fe and also to combine lighter nuclei. The former is called nuclear fission and the latter nuclear fusion. The energy released in both cases can and has been used both for power and nuclear weapons.

The small peaks showing disagreement are a result of quantum energy levels at particular values of Z and N and are the subject of the next lecture.