ORDINARY DIFFERENTIAL EQUATIONS

Variables

Changing entities are called variables

Independent and Dependent Variables

The variable to which values is assigned is called independent variables. The variable whose values obtained corresponding to assigned value is called dependent variable

For a function defined on a set and defined by $y = f(x)$. The variable x is the independent variable and the variable y takes the value corresponding to the assigned value of x, so y is a dependent variable.

Differential Equation

Equation express the relationship between dependent, independent variables and derivative of dependent variable with respect independent variable, is called Differential Equation.

Ordinary Differential Equation:

A differential equation which expresses relationship between one or more dependent variable which are function of a single independent variable and the total derivatives of dependent variables with respect to independent variable, i.e., the differential equation involving only total derivatives. That is "An ordinary differential equation-(ODE) contains differentials with respect to only one variable"

Partial Differential Equation:

A differential equation which contains the derivatives of one or more dependent variables with respect to more than one independent variables i.e., the differential equation involving partial derivatives. That is "Partial differential equations (PDE) contain differentials with respect to several independent variables".

Classification of Ordinary Differential Equations:

Simple/ordinary differential Equations:

An ordinary differential equation (ODE)which contains one dependent variable and derivatives of dependent variable with respect to independent variable.

Example

Let $y = y(x), x$ is independent variables then $\frac{d^3y}{dx^3} + sin\ x\frac{dy}{dx} = cos\ y$ is a simple / ordinary differential equation.

ENTRI

System of Differential Equations:

A differential equation which contains one independent variable, more than one dependent variables and derivatives of dependent variables with respect to independent variables.

Example

let $y = y(x)$ and $z = z(x)$ are two dependent variables of a single independent variable x then $\frac{dz}{dx} + \frac{dy}{dx} = sin\ x$ and $\frac{dy}{dx} + x\frac{dz}{dx} = cos\ x$ together is a system of the differential equations.

Order and Degree of Differential Equation

Order of the highest ordered derivatives occur in Differential Equation is called order of Differential Equation.

- Degree is the highest exponent of the highest derivative occur in it.
- Order of Differential Equation always exist but Degree need not exist always

Examples

- $(y'')^3 = (y'')^{\frac{4}{3}}$ order is 2 and degree is 9
- $log\ log\ (1 + y') = k$ order is 1 and degree not exist
- $y = sin\ sin\ \left(\frac{dy}{dx}\right)$ order is 1 and degree not exist
- $y'' + \sqrt{y'} = x$ order is 2 and degree is 2

Family of Curves

A family of curves is set of curves each of which is given by a function or parametrisation in which one or more parameters are variable.

Example

$A = \{(x, y): x^2 + y^2 = r^2\}$ represents the equation of family of circles with center $(0,0)$ and radius r

$B = \{(x, y): y = mx\}$ represents the equation of family of straight lines passing through origin

Solution of a Differential Equation

Let $F(x, y, y', y'', \ldots, y^n) = 0$ be n^{th} order differential equation then a real or complex valued function $\phi(x)$ of real variable x is solution if

- $\phi(x), \phi'^{(x)}, \ldots, \phi^{(n)}(x)$ where $\phi^{(k)}(x)$ is k^{th} derivative of $\phi(x)$
- $\phi(x)$ satisfies the differential equation $F\left(x, \phi(x), \phi'^{(x)}, \ldots, \phi^{(n)}(x)\right) = 0$

Classification of Solutions

Let $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$, be n^{th} order Differential Equation.

General Solution

Any solution of Differential Equation contains n independent arbitrary constants is General Solution.

Independence of Arbitrary Constants

Arbitrary constants operating in general solution must be independent and should not be reducible.

Particular solution

Any solution which can be obtained from general solution by taking some particular value of arbitrary constants

Singular Solutions

A singular solution is given as a tangent to every solution from a family of solutions.

First degree differential equation

The most general form of a first order first degree Differential Equation is $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is function of two variables defined in xy plane or it can be written as

$$M(x, y)dx + N(x, y)dy = 0$$

$M(x, y)$ and $N(x, y)$ are functions of two variables

Example

- $\frac{dy}{dx} = y + sinx$
- $(1 + x^2)\, dy - xy\, dx = 0$
- $\frac{dy}{dx} + \frac{2y}{x} = 6x^3$

Separable Differential Equation

A differential equation of the form $f(x, y) = f(x)\, g(x)$ is called separable form of Differential Equation with separation of variables.

It can be express as follows $f_1(x)\, dx = f_2(y)\, dy$

The solution is obtained by $\int f_1(x)\, dx = \int f_2(y)\, dy + c$

Example

$(1 + x^2)dy = (1 + y^2)dx.$

The given equation can be written as $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$.

Then integrating both sides,

we have $tan^{-1}(y) - tan^{-1}(x) = tan^{-1}(c)$, where c is an arbitrary constant.

ENTRI

Equation Reducible into Separable Form

If the first order Differential Equation is of the form $\frac{dy}{dx} = f(ax + by + c)$, it can be reducible into separable form in variable u, x by transforming

$$u = ax + by + c$$

Corresponding reduced Differential Equation is $a + bf(u) = \frac{du}{dx}$

Solution is $\int \frac{du}{a+bf(u)} = \int dx + c$

Homogenous Function

Let $S \subseteq R^2$ be subset of R^2. A function $f: S \to R$ is said to be homogenous function of degree n if $\forall\ \lambda > 0$

$$f(\lambda x. \lambda y) = \lambda^n f(x, y)$$

Homogenous Differential Equation

The first order Differential Equation $\frac{dy}{dx} = f(x, y)$ is called homogenous Differential Equation if $f(x, y)$ is homogenous function of degree 0.

That is $f(x, y)$ can be expressed as $f(x, y) = \phi\left(\frac{y}{x}\right)$ or $f(x, y) = \phi\left(\frac{x}{y}\right)$, where ϕ is arbitrary function. The Differential Equation can be reducible into separable form in new variables x, u using transformation $y = ux$.

Example

$(x + y)(dx - dy) = dx + dy.$

Re-writing the given equation, we get

$(x + y - 1)dx = (x + y + 1)dy$.

$or\ \frac{dy}{dx} = \frac{x+y-1}{x+y+1}.$

Let $x + y = v$ ... (2)

(2) $\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$ so that $\frac{dy}{dx} = \left(\frac{dv}{dx}\right) - 1.$

$\frac{dv}{dx} - 1 = \frac{v-1}{v+1}$ or $\frac{dv}{dx} = \frac{2v}{v+1}$ or $2dx = \left(1 + \frac{1}{v}\right) dv.$

$\therefore$ Integrating, $2x + c = v + log\ \ v$ or $x - y + c = log\ (x + y), C$ being an arbitrary constant

Differential Equation Reducible into Homogenous

Consider equation $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $\frac{a}{a'} \neq \frac{b}{b'}$

If c and c' both are zero the it is homogenous

If c and c' is not zero, then we change the variables

$x = X + h$ and $y = Y + k$

$$\therefore \frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'}$$

Or $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$

Choose h and k such that

$ah + bk + c = 0$ and $a'h + b'k + c' = 0$

$$\therefore \frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

Now follow step of homogenous DE

When $\frac{a}{a'} = \frac{b}{b'} = k$, the DE will be of $\frac{dy}{dx} = \frac{ax+by+c}{akx+bky+c'}$ form

In this case Differential Equation is solve by $v = ax + by$

Differentiating,

$$\frac{dv}{dx} = a + b\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} - a = b\frac{dy}{dx}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{b}\left[\frac{dv}{dx} - a\right]$$

$$\Longrightarrow \frac{v+c}{kv+c} = \frac{1}{b}\left[\frac{dv}{dx} - a\right]$$

$$\frac{dv}{dx} = a + \frac{b(v+c)}{kv+c}$$

Now equation is separable and solved.

Methods to Solve a First Order Differential Equation

Case I: Variable Separable

A first order differential Equation $y' = f(x, y)$ is called variable separable if the function $f(x, y)$ can be factored into the product of two functions of x and y.

$$f(x,y) = p(x)h(y)$$

$$\therefore \frac{dy}{dx} = p(x)h(y)\ \frac{dy}{h(y)} = p(x)dx$$

Note: $h(y) \neq 0$, if ∃ a y_0 such that $h(y_0) = 0$, then this y_0 will also be a solution of the differential equation.

Solution of the differential equation in (1) is given by:

$$\int \frac{dy}{h(y)} = \int p(x)dx + Q$$

Example

$$\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} + y\,sin\,x = 0$$

Case II: First Order Linear Differential Equation

General form of first order first degree linear DE is

$\frac{dy}{dx} + p(x)y = (x)$ P and Q are constants of functions of x

Homogeneous Linear Differential equation:

$$y' + p(x)y = 0$$

(a) Solution is by using integrating factor(IF).

Here IF $= e^{\int p(x)dx}$

Solution is given by

$$y = Ce^{-\int p(x)dx}$$

or

$$y.IF = C$$

Example:

$$y' + sinx\ \ y = 0$$

(b) Non-Homogeneous Linear Differential Equation:

$$y' + p(x)y = q(x)\ IF = e^{\int p(x)dx}$$

Solution is given by

$$ye^{\int Pdx} = Qe^{\int Pdx}dx + C$$

Or

$$y.IF = \int Q.IFdx + C$$

Example

$y' + 2xy = xe^{-x^2}$

$y' - y\, tan\, x = e^{sinx}$

$$3y' + y = 2e^{-x}$$

First order first degree Linear Differential Equation

General form of first order first degree linear DE is

$\frac{dy}{dx} + Py = Q$ P and Q are constants of functions of x

$e^{\int Pdx}$ is an integrating factor

Solution is given by

$$ye^{\int Pdx} = Qe^{\int Pdx}dx + C$$

Or

$$y.IF = \int Q.IFdx + C$$

Or,

General form of first order first degree linear DE is

$\frac{dx}{dy} + Px = Q$ P and Q are constants of functions of y

$e^{\int Pdy}$ is an integrating factor

Solution is given by

$$xe^{\int Pdy} = Qe^{\int Pdy}dy + C$$

Or

$$y.IF = \int Q.IFdy + C$$

Example

- $y'(x + y^2) = y$

  The equation can be written as

$$\Rightarrow y\frac{dx}{dy} - x = y^2\frac{dx}{dy} - \frac{x}{y} = y$$

  Integrating factor is given by

IF $= e^{\int \frac{1}{y} dy} = \frac{1}{y}$

$$x \cdot \frac{1}{y} = \int \frac{1}{y} y dy + c \frac{x}{y} = y + c$$

Hence the solution is given by

$$x = y^2 + cy$$

Equation reducible to Linear form

- An equation of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$ can be reduced to linear form by putting $f(y) = v$ so that equation becomes $\frac{dv}{dx} = Pv + Q$

  Solution be $v.e^{\int Pdx} = \int Qe^{\int Pdx} dx + C$
- An equation of the form $f'(x)\frac{dx}{dy} + P_1 f(x) = Q_1$

  P_1 and Q_1 are functions of y. It can be reduced to linear form by putting $f(x) = v$. So the equation becomes $\frac{dv}{dx} + P_1 v = Q_1$

  Solution be $v.e^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} dy + C$

Case III: Bernoulli's Equation

An equation of the form

$$y' + p(x)y = q(x)y^k, \qquad (1)$$

where k is an integer is called Bernoulli's Equation.
When $k = 0$ or 1 then DE (1) is linear.
When $k \neq 0$ or 1 , then (1is non-linear.
Standard form of (1) is obtained by letting $z = y^{1-k}$ and is given by

$$\frac{dz}{dx} + (1-k)p(x)z = (1-k)q(x) \qquad (2)$$

In order to solve (1) we need to solve the standard form (2) as in Case II.

Example

$y' + 2y = y^2$.

$$y^2.y' = x + y^3$$

Case IV: Homogeneous function

A function $f(x, y)$ is said to be homogeneous if it can be written as

$$f(x, y) = d(y/x)$$

A differential equation of the form

$$y' = f(x,y)$$

where $f(x,y)$ is homogeneous is called a first order homogeneous differential equation.
To solve this type of differential equations we substitute $y = Vx$ and $y' = V + x\frac{dV}{dx}$ in $y' = f(x,y)$

Then we get a variable separable DE, which can be solved as in Case I and to get final solution replace V by y/x.

Example

$$y' = \frac{x+y}{x-y}$$

$$y' = \frac{x^2+xy+y^2}{x^2}$$

Case V: Exact Differential Equation

$M(x,y)dx + N(x,y)dy = 0$ in some domain D is called an exact differential equation if there exists some function $f(x,y)$ such that $df = Mdx + Ndy$ in D.

Example:

$ydx + xdy = 0.$

- Let D be a rectangular domain and let first partial derivative of M and N be continuous in D. Then $M(x,y)dx + N(x,y)dy = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution of Exact Differential Equation:

$$\int_{y=\,constant} Mdx + \int (\,terms\ of\ N\ not\ conatinig\ x)dy = C$$

where C is the arbitrary constant.

Non Exact Differential Equation:

For DE $M(x,y)dx + N(x,y)dy = 0$ if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then DE is said to be non-exact.
Non Exact DE can be made exact by finding integrating factor.

Example

$ydx - xdy = 0$

$$(x^2 + xy)dx + xydy = 0$$

Integrating Factor

If μ is such that $\mu M(x,y)dx + \mu N(x,y)dy = 0$ becomes exact, then $\mu = \mu(x,y)$ is called an integrating factor.

Methods to find integrating factor:

Case 1

If $\frac{M_y - N_x}{N}$ is a function of x alone, say $f(x)$, then $IF = e^{\int f(x)dx}$

Case 2

If $\frac{N_x - M_y}{M}$ is a function of y alone, say $g(y)$, then $IF = e^{\int g(y)dx}$

Case 3

If the giver differential equation is homogenous with $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an IF.

Case 4

If the given differential equation is of the form $f_1(x,y)ydx + f_2(x,y)xdy = 0$ with $Mx - Ny \neq 0$, then $IF = \frac{1}{Mx-Ny}$

Case VI: Method of Successive approximation-Picard's Iteration

Theorem: $\phi(x)$ is a solution of the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ on interval I iff it is a continuous solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt$$

PICARD'S ITERATION

Consider the IVP

$$y' = f(x,y) \; and \; y(x_0) = y_0$$

Let $\phi(x)$ be a solution of the IVP, with $\phi(x_0) = y_0 = \phi_0$. Then Picards's nth iteration is given by

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt$$

By Picard's iteration we get a sequence of functions $\{\phi_r(x)\}$ which converges to a function $\phi(x)$ which satisfies the integral equation.

Existence and Uniqueness Theorems

Existence and Uniqueness Theorem for 1st order linear IVP If the functions $p(x)$ and $q(x)$ are continons on an open interval I, $\alpha < x < \beta$, then the IVP $y' + p(x)y = q(x), y(x_0) = y_0; x_0 \in I$, has a unique solution on I

Interval of Validity of a solution

The interval of validity of an IVP is the largest interval containing the point x_0 (initial point) in which the solution is defined.

Lipschitz Continuous

Let $f(x, y$ be function defined in a region $D \subset R^2$. Then f is said to be Lipschitz continuous in a variable y (dependent) if $\exists$ a $k > 0$ such that,

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$ and k is called Lipschitz constant.

Example

Let $R = \{(x, y) \in R^2 : |x - x_0| \leq a, |y - y_0| \leq b\}$ be a rectangular region. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then $f(x, y)$ is Lipschitz continuous.

If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on $D \subset R^2$ and $\left|\frac{\partial f}{\partial y}\right| \leq M$ on D. Then f is Lipschitz continuous in$y$.

Lipschitz constant $k = Sup \left|\frac{\partial f}{\partial y}\right|$

Peano's Theorem (Existence Theorem for general first order ODEs)

Let $R = \{(x, y) \in R^2 : |x - x_0| \leq a, |y - y_0| \leq b\}$. If the function $f(x, y)$ is continous on R. Then the IVP

$$y' = f(x, y), y(x_0) = y_0$$

has atleast one solution on the interval $|x - x_0| \leq h$ where $h = min\{a, b/M\}, |f(x, y)| \leq M$. That is if $f(x, y)$ is continuous on R then DE had atleast one solution in a neighbourhood of x_0.

Picard's Uniqueness Theorem (for general first order ODEs)

If $f(x, y)$ is continuous on R and satisfies Lipschitz condition, then the IVP $y' = f(x, y), y(x_0) = y_0$ has a unique solution on the interval $I: |x - x_0| \leq h$ where $h = min\{a, b/M\}, |f(x, y)| \leq M$.

Singular Solutions of first order ODEs

Types of solutions:

General solution

Solution involving arbitrary constant.

Particular Solution

Solution obtained by giving a particular value to the arbitrary constant.

Singular Solution

Sometimes the DE possess a solution which does not contain any arbitrary constant and which cannot be obtained from general solution by a assigning any particular value to the arbitrary constant. Such solutions are called singular solutions or integrals.

Envelope

It is a curve which touches each member of one parameter family of curves and at each point of which is touched by some member of the family.

Whenever the family of curves $\phi(x,y,c) = 0$ represented by the $DEF(x,y,p) = 0$ where $p = \frac{dy}{dx} = y'$ posses an envelope, the equation of the envelope can be the singular solution of the DE.

Methods to find Singular solution:

- finding envelope of the DE.
- finding p-discriminant or c-discriminant.

To find p-discriminant:

Let $F(x,y,p)$ be the given DE. Then the p-discriminant is obtained by eliminating p using the system of equations:

$$F(x,y,p) \quad = 0 \ \frac{\partial F}{\partial p} \quad = 0$$

To find c-discriminant:

Let $\phi(x,y,c)$ be the given DE. Then the c-discriminant is obtained by eliminating c using the system of equations:

$$\phi(x,y,c) \quad = 0 \ \frac{\partial \phi}{\partial c} \quad = 0$$

- If p and c discriminant have common factors, then that common factor will be the singular solution.

  Example:

  $y' = \sqrt{y}$

Clairaut's Equation:

It is differential equation of the form

$$y = px + f(p); p = y'$$

where f is continuously differentiable.
General solution of Clairaut's equation is:

$$y = cx + f(c)$$

where c is an arbitrary constant.

Example:

$$y = px + p^2$$

PARTIAL DIFFERENTIAL EQUATION

PDE

An equation involving one or Partial derivatives of an unknown function of 2 or more independent variables is known as Partial Differential Equation.

ORDER

Order of the highest partial derivative

DEGREE

Degree of the highest P.D after the equation has been made free from radicals or fractions

Notations

$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2},$

$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$

Classification of First Order PDE

1) Linear PDE

A first order PDE $f(x, y, z, p, q) = 0$ is said to be linear equation if it is linear in p, q and z

Given equation is of the form

$$P(x,y)\,p + Q(x,y)\,q = R(x,y)\,z + S(x,y)$$

2) Semi-Linear PDE

A first order PDE $f(x,y,z,p,q) = 0$ is said to be Semi linear equation if it is linear in p and q and is of the form

$$P(x,y)p + Q(x,y)\,q = R(x,y,z)$$

- Linear PDE $\Longrightarrow$ Semi-Linear PDE

3) Quasi Linear PDE

A first order PDE $f(x,y,z,p,q) = 0$ is said to be Quasi linear equation if it is linear in p and q and is of the form

$$P(x,y,z)p + Q(x,y,z)\,q = R(x,y,z)$$

- Semi Linear PDE $\Longrightarrow$ Quasi Linear PDE

4) Non Linear PDE

A first order PDE which is not Quasi Linear is called Non-linear PDE

i.e, it is not of any of the above 3 forms

Formation of PDE

- By elimination of arbitrary constants
- By elimination of arbitrary functions

SOLUTION OF 1st ORDER PDE

Consider the first order PDE $F(x,y,z,p,q) = 0$, Solution or Integral is a continuously differentiable function in a domain $D \subseteq R^2$ which satisfies the PDE $F(x,y,z,p,q) = 0$ in D

A two parameter family of solutions $g(x,y,z,a,b) = 0$, where a, b are arbitrary constants, of the PDE $F(x,y,z,p,q) = 0$ is called Complete solution\Complete integral

GENERAL SOLUTION

A solution of the form $\phi(u,v) = 0$, where ϕ is an arbitrary function and u, v are functions of x, y, z

$u = u(x,y,z),\ \ v = v(x,y,z)$

LAGRANGE'S METHOD FOR 1st OREDER PDE

For first order Quasi linear PDE

$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\,\frac{\partial z}{\partial y} = R(x,y,z)$

$Pp + Qq = R,$

Rule 1

Consider the Lagrange's Auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Is called equation of characteristics

Rule 2

Solving the equation of characteristics we get two independent solutions

$$u(x, y, z) = c_1$$

$$v(x, y, z) = c_2$$

These solutions represents two family of surfaces.

The intersection of these surfaces are called characteristic curves

Rule 3

We have $u = c_1 = \phi(c) = \phi(v)$

General solution of the form

$u = \phi(v)$ or $v = \phi(u)$ or $\phi(u, v) = 0$

Rule 4

Let P, Q, R be functions of x, y and z. Then each fraction of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$$

If $P_1 P + Q_1 Q + R_1 R = 0$, then $P_1 dx + Q_1 dy + R_1 dz = 0$

From this we get an integral where P_1, Q_1, R_1 are suitable functions of x, y and z and are called Multipliers.

- Usually P_1, Q_1, R_1 can be constant functions, (x, y, z),$\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$, $\left(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}\right)$, …

Rule 5

Let P, Q, R be functions of x, y and z. Then each fraction of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow (1)$$

Will be equal to

$$\frac{P_1dx + Q_1dy + R_1dz}{P_1P + Q_1Q + R_1R} \to (2)$$

Suppose the numerator of (2) is exact differential of the denominator of (2) then equation (2) can be combined with a suitable function of equation (1)

CAUCHY PROBLEM

To find an integral surface passing through the given curve (in a neighborhood of a curve C, such that the solution $z = z(x, y)$ takes a prescribed value $z_0(t)$ on C)

The curve C is called initial curve of the problem, and $z_0(t)$ is called the initial data or the Cauchy Data

Since solutions to PDE are surfaces called Integral Surfaces

Consider the PDE $Pp + Qq = R \ \to (1)$

Lagrange's Auxiliary Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \to (2)$$

Solving (2) we get two solutions

$$u = u(x, y, z) = c_1 \ v = v(x, y, z) = c_2 \} \to (3)$$

Let $x = x(t)\ y = y(t)\ z = z(t)$ be the parametric equation of the given curve using these equations in (3) we get

$$v\big(x(t), y(t), z(t)\big) = c_2 \ u\big(x(t), y(t), z(t)\big) = c_1 \} \to (4)$$

Eliminate the parameter `t' from (4) we get an equation of the form

$$g(c_1, c_2) = 0$$

and substitute the values of c_1 and c_2 from (3) Implies the required integral surface

UNIQUENESS OF CAUCHY PROBLEM

$Pp + Qq = R$, Quasi Linear PDE where, P, Q, R are functions of x, y, z in terms of $x(t), y(t), z(t)$

Then, has

- Unique Solution if
  i. $\frac{P}{\frac{dx}{dt}} \neq \frac{Q}{\frac{dy}{dt}} \neq \frac{R}{\frac{dz}{dt}}$, where P, Q, R are in terms of $x(t), y(t), z(t)$
  ii. $\frac{P}{\frac{dx}{dt}} \neq \frac{Q}{\frac{dy}{dt}} = \frac{R}{\frac{dz}{dt}}$
  iii. $\begin{vmatrix} P\ Q\ \frac{dx}{dt}\ \frac{dy}{dt} \end{vmatrix} \neq 0$
- Infinitely Many Solution

- $\frac{P}{\frac{dx}{dt}} = \frac{Q}{\frac{dy}{dt}} = \frac{R}{\frac{dz}{dt}}$
- No Solution
- $\frac{P}{\frac{dx}{dt}} = \frac{Q}{\frac{dy}{dt}} = a$ and $\frac{R}{\frac{dz}{dt}} \neq a$

CHARPITZ METHOD (For non linear PDE)

Consider the first order PDE $F(x, y, z, p, q) = 0$
Charpitz auxiliary equation

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dx}{pF_p + qF_q} = \frac{dp}{-F_x - pF_z} = \frac{dq}{-F_y - qF_z}$$

Solving these equation, find the values of p and q.
Then the complete integral is obtained from the equation

$$dz = pdx + qdy$$

by integration, we get complete integral of the form $g(x, y, z, a, b) = 0$ where a and b are arbitrary constants.

Special Methods for solving Charpitz equation

Type-1:

$F(p, q) = 0.$

Put $p = a$, a constant

Then $q = \phi(a)$

Then the complete integral will be

$dz = p\ dx + q\ dy$ becomes

$dz = a\ dx + \phi(a)\ dy$, on integrating we get

$$z = ax + \phi(a)y + b$$

Type 2: Equation not involving independent variable

$$F(z, p, q) = 0$$

Put $p = aq$, a constant

We get $q = \phi(a, z)$ and $p = a\ \phi(a, z)$

Then solution will be

$dz = p\ dx + q\ dy$ becomes

$dz = a\phi(a, z)\ dz + \phi(a, z)\ dy$

$\frac{dz}{\phi(a,z)} = a\ dx + dy$, on integrating we get the complete integral

Type 3: Separable Equation

Equations of the form $f(x, p) = g(y, q)$, (equations which z is absent and the terms containing x and p can be separable from those containing y and q

Let $f(x, p) = g(y, q) = a$

From $f(x, p) = a$, solve for p, and get $p = \phi(x, a)$

From $g(y, q) = a$, solve for q, and get $q = \psi(x, a)$

Then

$dz = p\, dx + q\, dy$ becomes

$dz = \phi(x, a)\, dx + \psi(x, a)\, dy$, on integrating ,

$$z = \int \phi(x, a)\, dx + \psi(x, a)\, dy + b$$

Is the complete integral

SECOND ORDER PDE

The general second order linear PDE is of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Where the coefficients A, B, C, D, F and G are in general functions of the independent variables x, y but do not depends on the unknown function u

Classification of Second Order PDE

Classification of second order PDE depends on the form of the leading part of the equations consisting of the second order terms. So for simplicity we can use the following general form

$Au_{xx} + Bu_{xy} + Cu_{yy} + I(x, y, u, u_x, u_y) = 0 \to (1)$

Then the second order PDE can be classified into 3

Hyperbolic, Parabolic and Elliptic

HYPERBOLIC

is said to be hyperbolic in some domain $D \subseteq R^2$ if

$$B^2 - 4AC > 0\ \forall x, y \in D$$

PARABOLIC

is said to be hyperbolic in some domain $D \subseteq R^2$ if

$$B^2 - 4AC = 0\ \forall x, y \in D$$

ELLIPTIC

is said to be hyperbolic in some domain $D \subseteq R^2$ if

$$B^2 - 4AC < 0 \; \forall x, y \in D$$

- Consider the equation $\frac{dy}{dx} = \frac{-B \pm \sqrt{B^2-4AC}}{2A} \rightarrow (2)$
  The real solution of (2) are called real characteristic curve of (1)
- Hyperbolic: $B^2 - 4AC > 0 \Rightarrow$ there are 2 real characteristic curves
- Parabolic: $B^2 - 4AC = 0 \Rightarrow$ there is one real characteristic curve
- Elliptic: $B^2 - 4AC < 0 \Rightarrow$ there is no real characteristic curve

Canonical Forms

Hyperbolic

$B^2 - 4AC > 0 \Rightarrow$ there are 2 real characteristic curves

$f_1(x, y) = c_1$and $f_2(x, y) = c_2$

To find the canonical or normal form we use the transformation

$\zeta = \zeta(x, y) = f_2(x, y)$ and $\eta = \eta(x, y) = f_1(x, y)$, provided

$|\zeta_x \; \zeta_y \; \eta_x \; \eta_y \,| \neq 0$

And its canonical form is of the form

$$u_{\zeta\zeta} - u_{\eta\eta} = g(\zeta, \eta, u, u_\zeta, u_\eta)$$

$$u_{\zeta\eta} = g(\zeta, \eta, u_\zeta, u_\eta)$$

Parabolic

$B^2 - 4AC = 0 \Rightarrow$ there is one real characteristic curve

$\frac{dy}{dx} = \frac{B}{2A}$, we get a solution $f(x, y) = c$

And the transformation is

$\zeta = \zeta(x, y) = f(x, y)$

$|\zeta_x \; \zeta_y \; \eta_x \; \eta_y \,| \neq 0$

And its canonical form is of the form

$$u_{\zeta\zeta} = g(\zeta, \eta, u_\zeta, u_\eta)$$

$$u_{\eta\eta} = g(\zeta, \eta, u, u_\zeta, u_\eta)$$

Elliptic

$B^2 - 4AC < 0 \Rightarrow$ there is no real characteristic curve

$\frac{dy}{dx} = \frac{B \pm i\sqrt{4AC-B^2}}{2A}$ we get complex solution

$\phi(x, y) = f_1(x, y) + i\, f_2(x, y)$

To find canonical form

$\zeta(x,y) = Re\ \phi$ and $\eta(x,y) = Im\ \phi$

$\left|\zeta_x\ \zeta_y\ \eta_x\ \eta_y\right| \neq 0$

And its canonical form is of the form

$$u_{\zeta\zeta} + u_{\eta\eta} = g(\zeta, \eta, u, u_\zeta, u_\eta)$$

Canonical Form of a Second order PDE

For second order linear PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

The canonical form is given by

$$\underline{A}u_{\zeta\zeta} + \underline{B}u_{\zeta\eta} + \underline{C}u_{\eta\eta} + \underline{D}u_\zeta + \underline{E}u_\eta + \underline{F}u = \underline{G}$$

Where

$$\underline{A} = A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2$$

$$\underline{B} = 2A\zeta_x\eta_x + B(\zeta_x\eta_y + \zeta_y\eta_x) + 2C\zeta_y\eta_y$$

$$\underline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$\underline{D} = A\zeta_{xx} + B\zeta_{xy} + C\zeta_{yy} + D\zeta_x + E\zeta_y$$

$$\underline{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$$\underline{F} = F$$

$$\underline{G} = G$$

HIGHER ORDER PDE WITH CONSTATN COEFFICIENTS

An equation of the form

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n\frac{\partial^n z}{\partial y^n} = F(x,y)$$

Where $a_0, a_1, \ldots, a_n$ are constants, is called a Liner Partial Differential Equation of n^{th} order with constant coefficients

Writing D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial y}$ then equation becomes

$$(a_0D^n + a_1D^{n-1}D' + a_2D^{n-2}D'^2 + \cdots + a_nD'^n)z = F(x,y)$$

Or

$$\phi(D, D')z = F(x,y)$$

The complete solution of the PDE consists of two parts

1) The Complimentary Function (CF), which is the complete solution of the equation $\phi(D, D')z = 0$
2) The particular integral (PI), which is a particular solution (free from arbitrary constants) of $\phi(D, D')z = F(x, y)$

Then the complete solution of the given PDE is

$$z = CF + PI$$

Rules for Finding CF

The auxiliary equation, put $D = m, D' = 1$ in

$$(a_0D^n + a_1D^{n-1}D' + a_2D^{n-2}D'^2 + \cdots + a_nD'^n)z = 0$$

Then it becomes

$$(a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n)z = 0$$

Case 1

If the roots of A.E are all different $m_1, m_2, \ldots m_n$

$$CF = f_1(y + m_1x) + f_2(y + m_2x) + \cdots + f_n(y + m_nx)$$

Case 2

If the roots of A.E are all equal $m_1, m_2, \ldots m_n = m$

$$CF = f_1(y + mx) + xf_2(y + mx) + \cdots + x^{n-1}f_n(y + mx)$$

General Method to Find PI

PI of the given PDE $\phi(D, D')z = F(x, y)$ is given by

$$PI = \frac{1}{\phi(D, D')} F(x, y)$$

Case 1

If $F(x, y) = e^{ax+by}$

$$PI = \frac{1}{\phi(a, b)} e^{ax+by}$$

If $\phi(a, b) = 0$, then

$$PI = \frac{x}{\phi'(a, b)} e^{ax+by}$$

Case 2

If $F(x, y) = sin\ sin\ (ax + by)$ or $cos\ cos\ (ax + by)$

$$PI = \frac{1}{\phi(D^2, DD', D'^2)} sin\ sin\ (ax + by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} sin\, sin\, (ax + by)$$

Case 3

If $F(x, y) = x^m y^n$

$$PI = \frac{1}{\phi(D, D')} x^m y^n = [\phi(D, D')]^{-1} x^m y^n$$

Where $[\phi(D, D')]^{-1}$ is the binomial expansion in powers $\frac{D'}{D}$ up to the degree n

Case 4

If $F(x, y) = e^{ax+by}\, v(x, y)$

$$PI = e^{ax+by} \frac{1}{\phi(D, D')} v(x, y)$$

$$= e^{ax+by} \frac{1}{\phi(D + a, D' + b)} v(x, y)$$

WAVE EQUATION

One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where $c^2 = \frac{T}{m}$, a constant

Subjected to the conditions

- Boundary condition: $u(0, t) = u(l, t) = 0$
- Initial condition : $u(x, 0) = f(x)$ and
  $\frac{\partial u}{\partial t} = 0$ at $t = 0$
  Where l is the length of the wire or string

D' Alembert Solution of Wave Equation

The solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad o \leq x \leq l, t > 0$$

Case 1

Subjected to

- Boundary condition: $u(0, t) = u(l, t) = 0$
- Initial condition: $u(x, 0) = f(x)$ and
  $\frac{\partial u}{\partial t}(x, 0) = 0$

Solution is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

Case 2

Subjected to

- Boundary condition: $u(0,t) = u(l,t) = 0$
- Initial condition: $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$

Solution is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

HEAT EQUATION (Diffusion Equation)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad \alpha > 0$$

Where α is the thermal diffusivity or conductivity

Subject to the

Boundary conditions: $u(0,t) = u(l,t) = 0$ and

Initial condition $u(x,0) = f(x)$, at $t = 0$

General Solution of Heat Equation

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\sin\left(\frac{n\pi x}{l}\right) e^{\frac{-n^2\pi^2\alpha t}{l^2}}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\sin\frac{n\pi x}{l}\, dx$$

LAPLACE EQUATION

A two dimensional Laplace equation is a second order PDE of the form

$$u_{xx} + u_{yy} = 0$$

- Laplace Equation is Elliptic

Solution by Method of Separation of Variable

Let the solution as $u = X(x)\,Y(y)$

$u_{xx} + u_{yy} = X''Y + XY'' = 0$

Let $\frac{X''}{X} = -\frac{Y''}{Y} = k$

Case 1

If $k = 0$

$$u(x,y) = (a_1x + a_2)(a_3x + a_4)$$

Case 2

If $k > 0$

$$u(x,y) = \left(b_1e^{\sqrt{k}x} + b_2e^{-\sqrt{k}x}\right)\left(b_3 \cos\cos\sqrt{k}y\ + b_4 \sin\sin\sqrt{k}y\ \right)$$

Case 3

If $k < 0$

$$u(x,y) = \left(c_1 \cos\cos\sqrt{-k}x\ + c_2 \sin\sin\sqrt{k}x\ \right)\left(c_3e^{\sqrt{-k}y} + c_4e^{-\sqrt{-k}y}\right)$$

Special Cases (Dirichlet Problem)

Case 1

$u_{xx} + u_{yy} = 0$ with

$u(x,0) = f(x),\ \ u(x,b) = 0,\ \ u(0,y) = 0,\ \ u(a,y) = 0$

Then the solution

$$u(x,y) = \sum_{n=1}^{\infty} \quad a_n \sin\sin\left(\frac{n\pi x}{a}\right) \sinh\sinh\left(\frac{n\pi(b-y)}{a}\right)$$

Where $a_n = \frac{2}{a}(\left(\frac{n\pi b}{a}\right) \int_0^a \quad f(x)\ \sin\sin\left(\frac{n\pi x}{a}\right)\ dx$

Case 2

$$u(x,0) = 0,\ \ u(x,b) = f(x),\ \ u(0,y) = 0,\ \ u(a,y) = 0$$

Then the solution

$$u(x,y) = \sum_{n=1}^{\infty} \quad b_n \sin\sin\left(\frac{n\pi x}{a}\right) \sinh\sinh\left(\frac{n\pi y}{a}\right)$$

Where $b_n = \frac{2}{a}(\left(\frac{n\pi b}{a}\right) \int_0^a \quad f(x)\ \sin\sin\left(\frac{n\pi x}{a}\right)\ dx$

Case 3

$$u(x,0) = 0,\ \ u(x,b) = 0,\ \ u(0,y) = g(y),\ \ u(a,y) = 0$$

Then the solution

$$u(x,y) = \sum_{n=1}^{\infty} \quad c_n \sin\sin\left(\frac{n\pi y}{b}\right) \sinh\sinh\left(\frac{n\pi(a-x)}{b}\right)$$

Where $c_n = \frac{2}{b}(\left(\frac{n\pi a}{b}\right) \int_0^b \quad g(y)\ \sin\sin\left(\frac{n\pi y}{b}\right)\ dy$

Case 4

$$u(x,0) = 0,\ \ u(x,b) = 0,\ \ u(0,y) = 0,\ \ u(a,y) = g(y)$$

Then the solution

$$u(x,y) = \sum_{n=1}^{\infty} \quad d_n \sin \sin \left(\frac{n\pi y}{b}\right) \sinh \sinh \left(\frac{n\pi x}{b}\right)$$

Where $d_n = \frac{2}{b}(\left(\frac{n\pi a}{b}\right) \int_0^b \quad g(y)\ \sin \sin \left(\frac{n\pi y}{b}\right)\ dy$