# FUNCTIONAL ANALYSIS

## NORMED SPACE

**Let $X$ be a linear space over $K$. A norm on $X$ is a function $\|.\|$ from $X$ to $R$ such that for all $x, y \in X$ and $k \in K$**

1. $\|x\| \geq 0$ , $\|x\| = 0$ iff $x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\|kx\| = |k|\|x\|$

### Examples:

**Euclidean Norm: let $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots x_n)$ with**

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- **Let $Y$ be a subspace of a normed space $X$. Then $Y$ and its closure $\overline{Y}$ are normed spaces with the induced norm.**

## Quotient Norm

**Let $Y$ be a closed subspace of a normed space $X$. For $x + Y$ in the quotient space $X/Y$, let**

$$|||x + Y||| = inf\{||x + y||: y \in Y\}$$

**Then $|||.|||$ is a norm on $X/Y$ called the quotient norm.**

- **Let $X$ be a normed space. The following are equivalent**
  - **(i) Every closed and bounded subset of $X$ is compact**
  - **(ii) The subset $\{x \in X: ||x|| \leq 1\}$ of $X$ is compact.**
  - **(iii) $X$ is finite dimensional.**

- **Let $||.||$ and $||.||'$ be norms on a linear space $X$. Then the norm $||.||$ is stronger than the norm $||.||'$ if and only if there is some $\alpha > 0$ such that $||x||' \leq \alpha||x||$ for all $x \in X$.**

Further, the norm $||.||$ is equivalent to the norm $||.||'$ if and only if there are $\alpha > 0$ and $\beta > 0$ such that $\beta||x|| \leq ||x||' \leq \alpha||x||$ for all $x \in X$.

## Continuity of Linear map

Let $X$ and $Y$ be normed spaces. If $X$ is finite dimensional, then every linear map from $X$ to $Y$ is continuous.

Conversely, if $X$ is infinite dimensional and $Y \neq \{0\}$, then there is a discontinuous linear map from $X$ to $Y$.

- Let $X$ and $Y$ be normed spaces and $F: X \to Y$ be a linear map such that the range $R(F)$ of $F$ is finite dimensional. Then $F$ is continuous if and only if the zero space $Z(F)$ of $F$ is closed in $X$.

## Bounded Linear Maps

A linear map from a normed space $X$ to a normed space $Y$ is continuous if and only if it maps bounded sets in $X$ onto bounded sets in $Y$. Such maps are known as bounded linear map.

The set of all such maps is denoted by $BL(X, Y)$.

## Linear Operator

A linear map from $X$ to itself is called an operator on $X$. Denoted by $BL(X)$.

## Bounded Linear Functional

A bounded linear map from $X$ to $K$. Denoted by $BL(X, K)$.

## Operator Norm

Let $X$ and $Y$ be normed spaces. For $F \in BL(X, Y)$, define

$$||F|| = sup\{||F(x)||: x \in X, ||x|| \leq 1\}$$

Then $||.||$ is a norm on $BL(X, Y)$, called the operator norm.

- Let $X$ be a normed space over $K$, and $f$ be a nonzero linear functional on $X$. If $E$ is an open subset of $X$, then $f(E)$ is an open subset of $K$.

## Hahn-Banach Separation theorem

Let $X$ be a normed space over $K$ and $E_1, E_2$ be nonempty disjoint convex subsets of $X$, where $E_1$ is open in $X$. Then there is a real hyperplane in $X$ which separates $E_1$ and $E_2$ in the following sense: For some $f \in X'$ and $t \in R$, we have

$$Ref(x_1) < t < Ref(x_2)$$

for all $x_1 \in E_1$ and $x_2 \in E_2$

## Hahn-Banach extension theorem

Let $X$ be a normed space over $K$, $Y$ be a subspace of $X$ and $g \in Y'$. Then there is some $f \in X'$ such that $f_Y = g$ and $||f|| = ||g||$

## Banach Space

A normed space $X$ over $K$ is called a Banach space if $X$ is complete in the metric $d(x, y) = ||x - y||$ induced by the norm $||.||$.

- A subspace $Y$ of a Banach space $X$ is a Banach space id and only if $Y$ is closed in $X$.
- A Banach space cannot have a denumerable basis.

## Schauder Basis

Let $X$ be a normed space. A countable subset $\{x_1, x_2, \ldots\}$ of $X$ is called a Schauder basis for $X$ if $||x_n|| = 1$ for each $n$ and if foe every $x \in X$, there are unique $k_1, k_2, \ldots$ in $K$ such that $x = \sum_n k_n x_n$.

ENTRI

## Uniform Boundedness Principle

**Let $X$ be a Banach space, $Y$ be a normed space and $\mathcal{F}$ be a subset of $BL(X,Y)$ such that for each $x \in X$, theset $\{F(x): F \in \mathcal{F}\}$ is bounded in $Y$. Then for each bounded subset $E$ of $X$, the set $\{F(x): x \in E, F \in \mathcal{F}\}$ is bounded in $Y$, that is $\mathcal{F}$ is uniformly bounded on $E$.**

**In particular, $sup\{||F||: F \in \mathcal{F}\} < \infty$.**

- **Geometrically, either each $F$ in $\mathcal{F}$ maps a given bounded subset of a Banach space $X$ into a fixed ball in the normed space $Y$,or else there is some $x \in X$ such that no ballin $Y$ contains all $F(x)$ with $F \in \mathcal{F}$.**

- **Let $X$ be a Banach space, $Y$ be a normed space and $(F_n)$ be a sequence in $BL(X,Y)$ such that the sequence $(F_n(x))$ converges in $Y$ for every $x \in X$. For $x \in X$, define**

$$F(x) = \lim_{n\to\infty} F_n(x)$$

## Banach-Steinhaus theorem

**$F$ is a bounded linear map from $X$ to $Y$ and**

$$||F|| \leq \lim_{n\to\infty} inf\ ||F_n|| \leq sup\{||F_n||: n = 1, 2, ..\} < \infty$$

- **Let $E$ be a totally bounded subset of $X$. Then $(F_n(x))$ converges to $F(x)$ uniformly for $x \in E$.**

## Resonance theorem

**Let $X$ be a normed space and $E$ be a subset of $X$. Then $E$ is bounded in $X$ if andonly if $f(E)$ is bounded in $K$ for every $f \in X'$.**

- **Let $X$ and $Y$ be normed spaces and $F: X \to Y$ be linear. Then $F$ is continuous if and only if $goF$ is continuous for every**

$$g \in Y'$$

## Continuous function:

Let $X$ and $Y$ be metric spaces and $F$ be a map from $X$ to $Y$. Then $F$ is continuous if $x_n \to x$ in $X$ implies that $F(x_n) \to F(x)$ in $Y$.

## Closed Map

A map $F$ is said to be closed if $x_n \to x$ in $X$ and $F(x_n) \to y$ in $Y$ imply that $y = F(x)$.

- A continuous map is closed.
- A closed map may not be continuous.
- If a closed map$F$ is bijective, then its inverse $F^{-1}$ is also a closed map.

## Closed graph theorem

Let $X$ and $Y$ be Banachspaces and $F: X \to Y$ be a closed linear map. Then $F$ is continuous.

### Projection:

A linear map $P$ from a linear space $X$ to itself is called a projection if $P^2 = P$.

- If $P$ is a projection so is $I - P$ and $R(P) = Z(I - P)$,

  $Z(P) = R(I - P)$.

  And follows that $X = R(P) + z(P)$ and $R(P) \cap Z(P) = \{0\}$.

- Let $X$ be a normed space and $P: X \to X$ be a projection. Then $P$ is a closed map if and only if the subspaces $R(P)$ and $Z(P)$ are closed in $X$. Further $P$ is continuous, if $X$ is a Banach space.

### Open Map

A map $F$ from a metric space $X$ to a metric space $Y$ issaid to be open if for every open set $E$ in $X$, its image $F(E)$ is open in $Y$.

- A map $F$ is continuous if and only if for every open set $E$ in $Y$, its inverse image $F^{-1}(E)$ is open in $X$.

- Let $X$ and $Y$ be a normed space and $F: X \to Y$ be linear. Then $F$ is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $||x|| \leq \gamma ||y||$.

## Open Mapping theorem

Let $X$ and $y$ be Banach spaces and $F: X \to Y$ be a linear map which is closed and surjective. Then $F$ is continuous and open.

# INNER PRODUCT SPACE

## Inner Product

Let $X$ be a linear space over $K$. An inner product on $X$ is a function $\langle\ ,\ \rangle$ from $X \times X$ to $K$ such that for all $x, y, z$ in $X$ and $k$ in $K$, we have

(i) Positive definiteness
$\langle x, y\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$

(ii) Linearity in the first variable
$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle kx, y\rangle = k\langle x, y\rangle$

(iii) Conjugate symmetry

$$\langle y, x\rangle = \overline{\langle x, y\rangle}$$

## Inner Product Space

An inner product space is a linear space with an inner product on it.

## Polarization identity

Let $\langle\ ,\ \rangle$ be an inner product on a linear space $X$. For all $x, y \in X$,

$$4\langle x, y\rangle = \langle x + y, x + y\rangle - \langle x - y, x - y\rangle + i\langle x + iy, x + iy\rangle - i\langle x - iy, x - iy\rangle$$

## Schwarz inequality

Let $\langle\ ,\ \rangle$ be an inner product on a linear space $X$. For all $x, y \in X$,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

The equality holds if and only if the set $\{x, y\}$ is linearly dependent.

- Let $\langle\ ,\ \rangle$ be an inner product on a linear space $X$. For $x \in X$, define $||x|| = \langle x, y\rangle^{\frac{1}{2}}$, the non-negative square root of $\langle x, x\rangle$. Then $|\langle x, y\rangle| \leq ||x||\ ||y||$ for all $x, y \in X$.

## Parallelogram Law

For all $x, y \in X$,

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$$

## Hilbert Space

An inner product space which is complete in the norm induced by the inner product is called a Hilbert space.

## Orthogonal Set

Let $X$ be an inner product space over $K$. For $x, y \in X$, we say that $x$ and $y$ are orthogonal if $\langle x, y\rangle = 0$.

Denoted as $x \perp y$.

- For subsets $E$ and $F$ of $X$, we write $E \perp F$ if $x \perp y$ for all $x \in E$ and $y \in F$.
- For a subset $E$ of $X$, we say $E$ is orthogonal if $x \perp y$ for all $x \neq y$ in $E$.

## Orthonormal Sets

An orthogonal set $E$ of $X$ in which $\langle x, x\rangle = 1$ for all $x \in E$ is called an orthonormal subset.

### Pythagoras theorem

Let $X$ be an inner product space. Let $\{x_1, x_2, x_3, \ldots x_n\}$ be an orthogonal set in $X$. Then

$$||x_1 + x_2 + x_3 + \cdots + x_n||^2 = ||x_1||^2 + ||x_2||^2 + \cdots + ||x_n||^2$$

## Bessel's Inequality

**Let $\{u_1, u_2, \dots\}$ be a countable orthonormal set in an inner product space $X$ and $x \in X$. Then**

$$\sum_n |\langle x, u_n\rangle|^2 \le ||x||^2$$

**Where the equality holds if and only if $x = \sum_n \langle x, u_n\rangle\, u_n$.**

## Orthonormal Basis

**An orthonormal set $\{u_\alpha\}$ in a Hilbert space $H$ is said to be an orthonormal Basis for $H$ if it is maximal in the sense that if $\{u_\alpha\}$ is contained in some orthonormal subset $E$ of $H$, then, in fact, $E = \{u_\alpha\}$.**

## Orthogonal complement

**Let $V$ be an inner product space and $W$ be any setoff vectors in $V$. The orthogonal complement of $W$ is the set $W^\perp$ of all vectors in $V$ which are orthogonal to every vectors in $W$.**

$$W^\perp = \{y \in V : y \perp x \,, \forall x \in W\}$$

- **$W^\perp$ is a closed subspace of $V$.**
- **$W^{\perp\perp} = W$**

## Dual

**The set $X'$ of all continuous linear functional on $X$ is called the dual of $X$.**

- **$X'$ is a linear space under point wise addition and scalar multiplication.**

## Riesz representation theorem

**Let $H$ be a Hilbert space and $f \in H'$. Then there is a unique $y \in H$ such that $f(x) = \langle x, y\rangle, x \in H$.**

## Unique Hahn-Banach Extension Theorem

**Let $H$ be a Hilbert space, $G$ be a subspace of $H$ and $g$ be a continuous linear functional on $G$. Then there is a unique continuous linear functional $f$ on $H$ such that $f_{|G} = g$ and $||f|| = ||g||$.**

## Weak Convergence and Weak Boundedness

**Let $(x_n)$ be a sequence in a Hilbert space $H$. We say that $(x_n)$ converges to $x$ in $H$ if $||x_n - x|| = \langle x_n - x, x_n - x\rangle^{\frac{1}{2}} \to 0$ as $n \to \infty$ and write $x_n \to x$.**

**We say $(x_n)$ is weak convergent (or converges weakly) to $x$ in $H$ if $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$ for every $y \in H$, and then write $x_n \xrightarrow{w} x$.**

- **A subset of a Hilbert space is weak bounded if and only if it is bounded.**

## Uniform Boundedness Principle

**Let $\mathcal{F}$ be a set of continuous linear functionals on $H$. Then $\{f(y): f \in \mathcal{F}\}$ is a bounded subset of $K$ for each $y \in H$ if and only if the set $||f||: f \in \mathcal{F}\}$ is bounded. This is known as uniform Boundedness principle for continuous linear functionals on $H$.**

# BOUNDED OPERATORS ON HILBERT SPACE

## Operator

An operator $A$ on an inner product space $X$ over $K$, is a linear map $A$ from $X$ to $X$.

## Bounded Operator

An operator is said to be bounded if $||A(x)|| \leq \alpha||x||$ for all $x \in X$ and some $\alpha > 0$, where $||x|| = \sqrt{\langle x, x \rangle}$.

Set of bounded operators on $X$ is denoted as $BL(X)$.

- A bounded operator is uniformly continuous on $X$.

## Invertible Bounded Operator

$A \in BL(X)$ is invertible if there is some $B \in BL(X)$ such that $AB = BA = I$, where $I$ is the identity operator on $X$.

- Let $H$ be a Hilbert space and $A \in BL(H)$. Then there is a unique $B \in BL(H)$ such that for all $x, , y \in H$,
$$\langle A(x), y \rangle = \langle x, B(y) \rangle$$

## Normal Operator

Let $A \in BL(H)$. Then $A$ is called normal if $A^*A = AA^*$

## Unitary Operator

Let $A \in BL(H)$. Then $A$ is called unitary if $A^*A = I = AA^*$

i.e $A^{-1} = A^*$

## Self Adjoint Operator

Let $A \in BL(H)$. Then $A$ is called self adjoint operator if $A^* = A$.

- $A$ unitary $\Rightarrow A$ normal
- A is self adjoint $\Rightarrow A$ is normal

- Let $H$ be a Hilbert space
  (a) Let $A$ and $B$ be self adjoint. Then $A - B$ is self adjoint. Also $AB$ is self adjoint if and only if $A$ and $B$ commute.
  (b) Let $A$ and $B$ be unitary. Then $AB$ is unitary.

## Positive Operator

A self adjoint operator $A$ on a Hilbert space $H$ over $K$ is said to be positive if $\langle A(X), x\rangle \geq 0$ for all $x \in H$.

- Orthogonal projection is a positive operator.

## Positive Definite

A self adjoint operator $A$ on a Hilbert space $H$ is said to be positive definite if $\langle A(x), x\rangle \geq 0$ for all non-zero $x \in H$