Virial theorem

Consider an isolated system of N objects . masses, coordinates and velocities are given $m_i, \vec{r}_i, \vec{V}_i$.

The relation between the total kinetic energy and the total potential energy. The energies are

$$K = \frac{1}{2} \sum_i m_i \ [\ \dot{x}_i^{\,2} \ + \ \dot{y}_i^{\,2} \ + \ \dot{z}_i^{\,2}\ ]$$

(1)

$$W = -\frac{1}{2} G \sum_{i,j} \frac{m_i \, m_j}{|\vec{r}_i - \vec{r}_j|}$$

We start with defining moment of inertia ;

$$I = \sum_{i=1}^{N} m_i \,[\ x_i^{\,2} \ + \ y_i^{\,2} \ + \ z_i^{\,2}]$$

(2)

We will differentiate I with respect to time t twice then use equations of motion.

At some moment we need to use Newton's law of gravity

$$\frac{d\,I}{d\,t} = 2 \sum m_i \ [x_i \dot{x}_i \ + \ y_i \dot{y}_i \ + \ z_i \dot{z}_i\ ]$$

(3)

The second derivative

$$\frac{d^2 I}{d\,t^2} = 2 \ \sum m_i \ [\dot{x}_i^{\,2} + \ \dot{y}^2{}_i \ + \ \dot{z}_i^{\,2} + \ddot{x}_i^{\,2} x_i + \ddot{y}_i^{\,2} y_i + \ddot{z}_i^{\,2} z_i]$$

(4)

Time to use the equations of motion.

$$\ddot{\vec{r}}_i = \ -G \ \sum_i \frac{m_i \, m_j}{|\vec{r}_i - \vec{r}_j| 3} \qquad (5)$$

Take the last three terms in equ (4) and substitute acceleration with their expressions in equations of motion.

$$\sum m_i\,[\,\dot{x}^2 x + \dot{y}^2\, y + \dot{z}^2 z\,] = -\,G \sum_i \frac{m_i\, m_j}{|\vec{r}_{ij}|3}\ [\,x_i{}^2 + y_i{}^2 + z_i{}^2 - x_i x_j - y_i y_j - z_i z_j]$$

(6)

Note that in this double sum the index i is independent of index j . As a matter of fact, we can use any letter, the result is same. In equation (7) we can swap the indexes i⟺ j . Thus

$$S = \sum m_i\,(\,x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i)$$

$$= \frac{-G}{2} \sum_{i\,,j} \frac{m_i\, m_j}{|\vec{r}_{ij}|3}\,[\,x_i{}^2 + y_i{}^2 + z_i{}^2 - x_i x_j - y_i y_j - z_i z_j] \; -$$

$$\frac{-G}{2} \sum_{i\,,j} \frac{m_i\, m_j}{|\vec{r}_{ij}|3}\,[\,x_j{}^2 + y_j{}^2 + z_j{}^2 - x_i x_j - y_i y_j - z_i z_j] \qquad (7)$$

Collect the terms with x_i ; $x_i{}^2$ - $2x_i x_j + x_j{}^2 = (x_i - x_j)^2$ and do the same for other components. We rewrite the sum S in much more compact way ;

$$S = \frac{-G}{2} \sum_{i\,,j} \frac{m_i\, m_j}{|\vec{r}_{ij}|3} = -\,\frac{G}{2} \sum_i \frac{m_i\, m_j}{|\vec{r}_i - \vec{r}_j|} = W \qquad (8)$$

Going back to equ (4) we get

$$\frac{d^2 I}{d\,t^2} = 2K + W \qquad (9)$$

This is virial theorem. Typically it is applied with the assumption that $\frac{d^2 I}{d\,t^2} = 0$

Implying a stationary system.