Phase space of a classical system

The microstate of a given classical system, at any time t, may be defined by a specifying the instantaneous position and momenta of all the particles constituting the system. Thus, if N is the number of particles in a system, the definition of a microstate requires the specification of 3N position coordinates $q_1, q_2, q_3, \ldots q_{3N}$ and 3N momentum coordinates $p_1, p_2, p_3, \ldots p_{3N}$. Geometrically, the set of coordinates (q_i, p_i) Where i = 1,2,3, ...3N, may be regarded as a point in a space of 6N dimensions. We refer to this space as the phase space, and the phase point (q_i, p_i) as a *representative point*, of the given system.

The coordinates q_i and p_i are the functions of time t, the precise manner in which they vary with t is determined by the cannonical equations of motion.

$$\dot{q_i} = \frac{\partial H(q_i, p_i)}{\partial P_i} \qquad (1)$$

$$\dot{p_i} = - \frac{\partial H(q_i, p_i)}{\partial P_i}$$

$$i = 1, 2, 3, \ldots$$

Where H (q_i, p_i) is the Hamiltonian of the system. As the time passes the set of coordinates (q_i, p_i) which also defines the microstate of the system, undergoes a continual change. Correspondingly, our representative point in the phase space carves out a trajectory whose direction at any time t, is determined by the velocity vector V≡ $(\dot{q}_1, \dot{p}_1)$ which in turn given by the equations of motion. It is not difficult to see that the trajectory of the representative point must remain within in a limited region of phase space; this is so because a finite volume V directly limits the value of the coordinates q_i, while a finite value of energy E limits both the values of q_i *and the* p_i. In particular, if the total energy of the system is known to have precise value, say E the corresponding trajectory will be restricted to the hypersurface

$$H(q_i, p_i) = E \qquad (2)$$

Of the phase space; on the other hand, if the total energy may lie anywhere in the range (E - $\frac{1}{2}\Delta$, E + $\frac{1}{2}\Delta$) , the corresponding trajectory will be restricted to the "hypershell" defined by these limits.

Now, if we consider an ensemble of systems then, at any time t, the various number of ensembles will be in all sorts of possible microstates indeed, each one of these microstate that is supposed to be common to all members of the ensemble. In the phase space, the corresponding picture will consist of of swarm of representative points, one for each member of the ensemble, all lying within the "allowed " of the space. As the time passes, every member of the ensemble undergoes a continual change of microstates; correspondingly; the representative points constituting the the swarm continually move along their respective trajectories. The overall picture of this movement posses some important pictures that are best illustrated in terms of what we call a density function ρ (q, p ; t). this function is such that at any time t, the number of representative points in the "volume element" $(d^{3N}qd^{3N}p)$around the point (q, p) of the phase space is given by the product ρ (q, p ; t)$(d^{3N}qd^{3N}p)$. .
clearly the density function symbolizes the manner in which the members of the ensemble are distributed over all possible microstates at different instants of time. Accordingly, the ensemble average $\langle f \rangle$ of a given physical quantity f(q, p) which may different for systems in different microstates would be given by

$$\langle f \rangle = \frac{\int f(q,p)\, \rho(q,p;t) d^{3N}q d^{3N}p}{\int \rho(q,p;t) d^{3N}q d^{3N}p} \qquad (3)$$

The integration in (3) extended over the whole of the phase space, however it is only the populated regions of the phase space ($\rho \neq 0$) that really contribute. In general, the ensemble average energy $\langle f \rangle$ may itself be a function of time.

An ensemble is said to be stationary if ρ does not depend explicitly on time, that is, at all times.

$$\frac{\partial \rho}{\partial t} = 0 \qquad (4)$$

Clearly, for such an ensemble the average value (f) of any physical quantity f(q,p) will be independent of time. Naturally, a stationary ensemble qualifies to represent a system in equilibrium.