Liouville's theorem and its consequences

Consider an arbitrary " volume " ω in the relevant region of the phase space and let the " surface " enclosing the volume be denoted by σ. then the rate at which the number of representative points in this volume increases with time is written as

$$\frac{\partial}{\partial t}\int_{\omega} \rho \, d\omega \qquad (1)$$

Where $d\omega = (d^{3N}q d^{3N}p)$. on the other hand, the net rate at which the representative points flow out of ω is given by

$$\int_{\sigma} \rho \, v \, . \hat{n} \, d\sigma \, ; \qquad (2)$$

Here, **v** is the velocity vector of the representative points in the region of the surface element dσ while $\hat{n}$ is the unit vector normal to this element. By the divergence theorem (2) can be written as

$$\int_{\omega} div \, (\rho \, v) \, d\omega \qquad (3)$$

The operation of divergence means

$$\text{div}(\rho v) \equiv \sum_{i=1}^{3N} \left\{ \frac{\partial}{\partial q_i}(\rho \dot{q}_i) + \frac{\partial}{\partial q_i}(\rho \dot{p}_i \right\} \qquad (4)$$

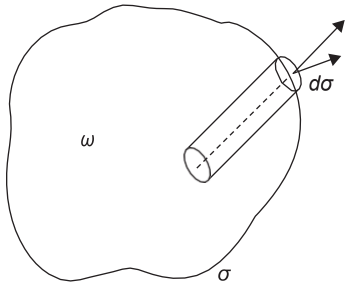

Figure : the hydrodynamics of the representative points on phase space.

In the view of fact that there are no " sources " or "sinks " in the phase space and hence the total number of representative points remains conserved. We have (1) and (3)

$$\frac{\partial}{\partial t}\int_{\omega} \rho \, d\omega = \int_{\omega} div\,(\rho\, v)\, d\omega$$

(5)

$$\int_{\omega} [div\,(\rho\, v) + \frac{\partial \rho}{\partial t}]\, d\omega = 0 \qquad (6)$$

Now the necessary and sufficient condition for the integral (6) to vanish for all arbitrary volumes ω is that the integrand itself vanish everywhere in the relevant region of the phase space. Thus we must have

$$\frac{\partial \rho}{\partial t} + div\,(\rho v) = 0 \qquad (7)$$

Which is the equation of continuity for the swarm of the representative points. Combining (4) and (7)

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3N} \left\{\frac{\partial}{\partial q_i}(\rho \dot{q}_i) + \frac{\partial}{\partial q_i}(\rho \dot{p}_i\right\} = 0$$

(8)

The last group of terms vanishes identically and since ρ = ρ (q, p, t), the remaining term in (8) may combine to form the total time derivative of ρ, with the result that.

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t} + [\rho\, , H\,] = 0 \qquad (10)$$

Equation (10) embodies Liouville's theorem. According to this theorem the local density of the representative points, as viewed by an observer moving with a representative point, says constant in time. Thus thee swarm of the representative points moves in the phase space in essentially the same manner as an incompressible fluid moves in the physical space.