Beta Function and its Applications

Abstract

The Beta function was rst studied by Euler and Legendre and was given its name by Jacques Binet. Just as the gamma function for integers describes factorials, the beta function can de ne a binomial coe¢ cient after adjusting indices.The beta function was the rst known scattering amplitude in string theory, rst conjectured by Gabriele Veneziano. It also occurs in the theory of the preferential attachment process, a type of stochastic urn process.The incomplete beta function is a generalization of the beta function that replaces the de nite integral of the beta function with an inde nite integral.The situation is analogous to the incomplete gamma function being a generalization of the gamma function.

1 Introduction

The beta function $(p;q)$ is the name used by Legendre and Whittaker and Watson(1990) for the beta integral (also called the Eulerian integral of the rst kind). It is de ned by

$$(p,q) = \frac{(p-1)!(q-1)}{(p+q-1)!}$$

To derive the integral representation of the beta function, we write the product of two factorial as

$$m!n! = \int_0^\infty e^{-u}u^m du \int_0^\infty e^{-v}v^n dv.$$

Now taking $u = x^2; v = y^2$;so

$$m!n! = 4 \;_0\; e^- \quad x^2 \;\; ^{+1}dx \;_0\; e^- \quad y^2 \;\; ^{+1}dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} |x|^{2m+1} |y|^{2n+1} dxdy.$$ R1 $u2$

m R1 $v2$ n

Transforming to polar coordinates with $x = r\cos; y = r\sin$

$m!n! =$ R02 R01 e^{v2} j$r\cos$ j^{2m+1} j$r\sin$ j^{2n+1} $rdrd$ to get

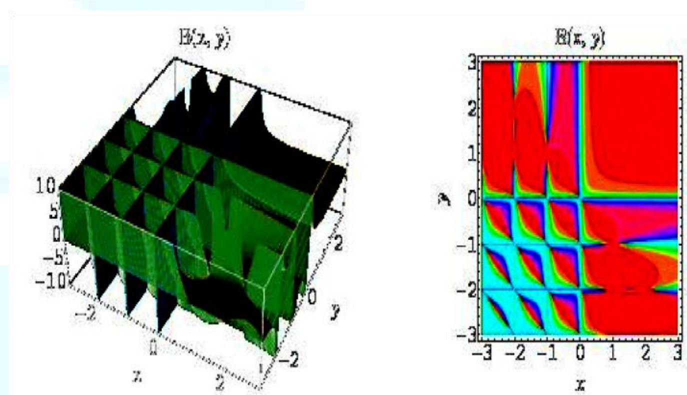
$m!n! = 2(m + n + 1)!$R0=2 $\cos^{2m+1} \sin^{2n+1} d$ The beta

$$(m+1,n+1) = 2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta$$

function is then de ned by

$$= \frac{m!n!}{(m+n+1)!}$$

Rewriting the arguments then gives the usual form for the beta function,



$$(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!(q-1)}{(p+q-1)!}$$

The general trigonometric form is

$$\int_0^{\pi/2} \sin^n x \cos^m x dx = \frac{1}{2}\beta\{\frac{1}{2}(n+1), \frac{1}{2}(m+1)\}$$

[1][2][5]

3-D Image of Beta Function

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1.1 *Beta Integral:-*

$$\beta_a(x) = \int_0^1 t^{a-1}(1-t)^{x-1}dt,$$

is called the Eulerian integral of the rst kind by Legendre and Whittaker and Watson(1990).The solution of this integral is the Beta function $(p+1;q+1)$: [1][2]

The Beta function evolves from Gamma function

$$(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!(q-1)}{(p+q-1)!}$$

where signi es the Gamma function.

Relationship between the Gamma function and the Beta function can be derived as

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u}u^{x-1}du \int_0^\infty e^{-v}v^{y-1}dv.$$

Now taking $u = a^2, v = b^2, so$ ∞ ∞

(x) (y) = 4R0 Re $a^2 a^{2x+1}{}_2\, da_2$ Rj0 j e b^2jb^2j$y+1$ db

= 1 1 e $(a+b)$ a 2x 1 b 2y 1 $dadb$:

R1 1

Transforming to polar coordinates with $a = r\cos;b= r\sin$

(x) $(_2y)$ R =02 R0^1 e^{r^2} j$r\cos$ j$^{2x\,1}$ j$r\sin$ j$^{2y\,1}$ $rdrd$

R

=

1 e r r2x+2y2rdr 2 $(\cos)$2x 1$(\sin)$2y 1 d

0 R0

= $(x+y)$ $(x;y)$

Hence, rewriting the arguments with the usual form of Beta function:

$$(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}:$$

A somewhat more straightforward derivation :

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^x \;\; {}^1e \;\; {}^tdt \int_0^\infty s^y \;\; {}^1e \;\; {}^sds$$

$$= \int_{t=0}^\infty \int_{s=0}^\infty t^{x-1}s^{y-1}e^{-(t+s)}dsdt$$

The argument in the exponential inspires us to employ the substitution

$= s + t$

$= t$

Thusj$J$j = 1; where J is the Jacobian of the transformation. Using this transformation,

$$\Gamma(x)\Gamma(y) = \int_{\sigma=0}^\infty \int_{\tau=0}^\sigma \tau^{x-1}(\sigma-\tau)^{y-1}e^{-\sigma}d\tau d\sigma$$

$$= \int_{\sigma=0}^\infty \int_{\tau=0}^\sigma \tau^{x-1}\sigma^{y-1}(1-\tfrac{\tau}{\sigma})^{y-1}e^{-\sigma}d\tau d\sigma$$

Again, now the comparison to

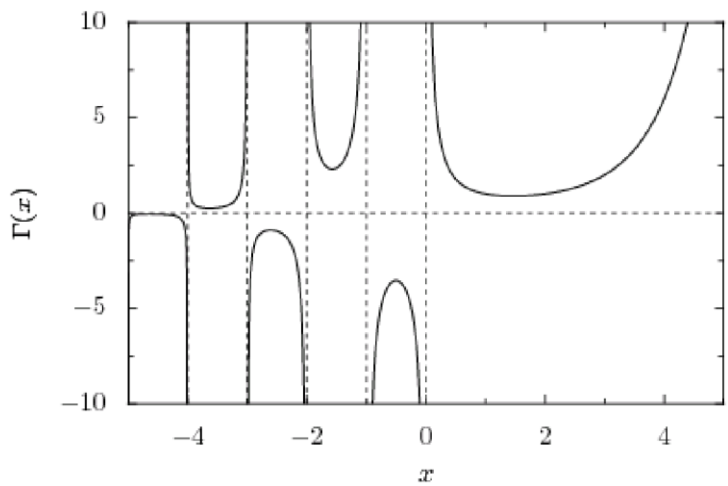
$(x;y)$ leads us to : r = _

;q = where the Jacobian is now:jJj

= q:

This leads to an easy identi cation with the expected result:

(x) (y) = Rq1=0RRr1=0 q $(rq)$$x$ 1Rq(y 1) (1 r)y 1 e $q$$drdq$



= Rq1=0R r1=0 $r$$x$ 1 (1 r)y 1 $q$$x$+$y$ 1e $q$$drdq$

= 01 $q$$x$+$y$ 1e $q$$dq$ 01 $r$$x$ 1 (1 r)y 1 dr

= $(x+y)$ $(x;y)$

As the gamma function is de ned as an integral, the beta function can similarly be de ned in the integral form:

$$\beta_a(x) = \int_0^1 t^{a-1}(1-t)^{x-1}dt.$$

Graph of Gamma Function

The trignometric form of Beta function is

$(x;y) = 2R_{0\_2} \sin^{2x\,1} \cos^{2y\,1} d$; R(x)>0, R(y)>0.

Putting it in a form which can be used to develop integral representations of the Bessel functions and hypergeometric function,

$$\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt,$$

R(x)>0, R(y)>0.

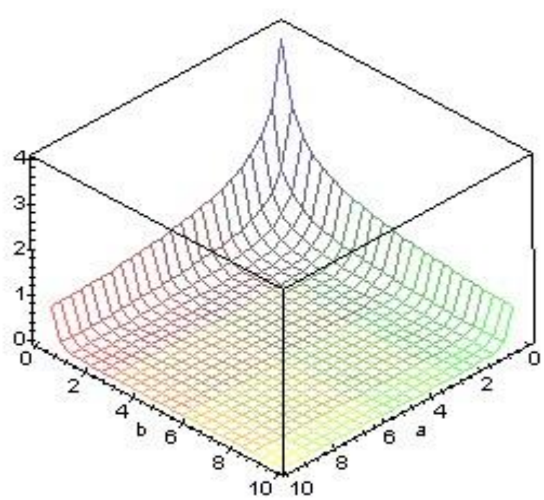
$$\beta(x,y) = \frac{1}{y}\sum_{n=0}^{\infty}(-1)^n \frac{(y)_{n+1}}{n!(x+n)}$$ where (x)

is the gamma function .The second identity shows in particular $\Gamma(\frac{1}{2}) = \sqrt{\ }$:

Just as the gamma function for integers describes factorials, the beta function can de ne binomial coe¢ cient after adjusting indices.

$$\binom{n}{k} = \frac{1}{(n+1)\beta(n-k+1,k+1)}$$

[2][5]



Graph of the Beta Function

2 Applications:-

2.1 *Beta function and String Theory:-*

The Beta function was the rst known Scattering amplitude in String theory, rst conjectured by Gabriele Veneziano ,an Italian theoretical physicist and a founder of string theory.

Gabriele Veneziano, a research fellow at CERN (a European particle accelerator lab) in 1968, observed a strange coincidence - many properties of the strong nuclear force are perfectly described by the Euler beta-function, an obscure formula devised for purely mathematical reasons two hundred years earlier by Leonhard Euler. In the urry of research that followed, Yoichiro Nambu of the University of Chicago, Holger Nielsen of the Niels Bohr Institute, and Leonard Susskind of Stanford University revealed that the nuclear interactions of elementary particles modeled as one-dimensional strings instead of zerodimensional particles were described exactly by the Euler beta-function. This was, in e⁄ect, the birth of string theory.

The Euler Beta function appeared in elementary particle physics as a model for the scattering amplitude in the so-called "dual resonance model". Introduced by Veneziano in the 1970th in order to t experimental data , it soon turned out that the basic physics behind this model is the string(instead of the zero-dimensional mass-point).[4]

2.2 *Preferential Attachment process:-*

Preferential Attachment to a class of processes in which some quantity, typically some form of wealth or credit, is distributed among a number of individuals or objects according to how much they already have, so that those who are already wealthy receive more than those who are not. The principal reason for scienti c interest in preferential attachment is that it can, under suitable circumstances, generate power law distributions of wealth[2].[3]

2.2.1 *Stochastic Urn Process and Beta Function:-*

A preferential attachment process is a stochastic urn process, meaning a process in which discrete units of wealth, usually called "balls", are added in a random or partly random fashion to a set of objects or containers, usually called "urns". A preferential attachment process is an urn process in which additional balls are added continuously to the system and are distributed among the urns as an increasing function of the number of balls the urns already have. In the most commonly studied examples, the number of urns also increases continuously, although this is not a necessary condition for preferential attachment and examples have been studied with constant or even decreasing numbers of urns.

$$P(k) = \frac{B(k+a,\gamma)}{B(k_0+a,\gamma\quad 1)}$$

Linear preferential attachment processes in which the number of urns increases are known to produce a distribution of balls over the urns following the socalled Yule distribution . In the most general form of the process, balls are added to the system at an overall rate of m new species for each new urn. Each Figure 1: Probability density Function

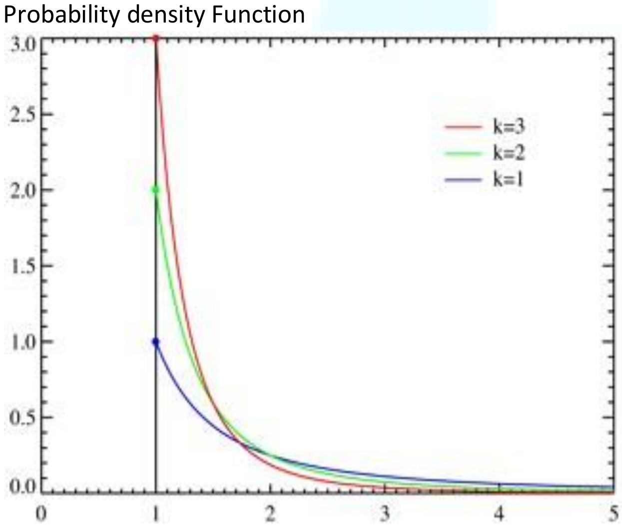


Figure 2: Pareto probability density functions for various k with $x_m = 1$.The horizontal axis is the x parameter.

newly created urn starts out with k_0 balls and further balls are added to urns at a rate proportional to the number k that they already have plus a constant $a > k_0$. With these de nitions, the fraction $P(k)$ of urns having k balls in the limit of long time is given by

$$P(k) = \frac{B(k+a,\gamma)}{B(k_0+a,\gamma-1)}$$

for k 0(and zero otherwise), where $(x;y)$ is the Euler beta function:

$$(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

with (x) being the standard gamma function, and

$$= 2 + \frac{k_{0+a}}{m}$$

In other words, the preferential attachment process generates a "long-tailed" distribution following a Pareto distribution or power law in its tail. This is the primary reason for the historical interest in preferential attachment. The species distribution and many other phenomena are observed empirically to follow power laws and the preferential attachment process is a leading candidate mechanism to explain this behavior. Preferential attachment is considered a possible candidate for, among other things, the distribution of the sizes of cities, the wealth of extremely wealthy individuals, the number of citations received by learned publications and the number of links to pages on the world wide web.[2][3]