1.15 DIRAC DELTA FUNCTION

From Example 1.6.1 and the development of Gauss' law in Section 1.14,

$$\int \nabla \cdot \nabla\left(\frac{1}{r}\right) d\tau = -\int \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) d\tau = \begin{cases} -4\pi \\ 0, \end{cases} \tag{1.169}$$

depending on whether or not the integration includes the origin $\mathbf{r} = 0$. This result may be conveniently expressed by introducing the Dirac delta function,

$$\boxed{\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r}) \equiv -4\pi\delta(x)\delta(y)\delta(z).} \tag{1.170}$$

This Dirac delta function is **defined** by its assigned properties

$$\delta(x) = 0, \qquad x \neq 0 \tag{1.171a}$$

$$\boxed{f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx,} \tag{1.171b}$$

where f(x) is any well-behaved function and the integration includes the origin. As a special case of Eq. (1.171b),

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1. \tag{1.171c}$$

From Eq. (1.171b), δ(x) must be an infinitely high, infinitely thin spike at x = 0, as in the description of an impulsive force (Section 15.9) or the charge density for a point charge.[1]The problem is that **no such function exists**, in the usual sense of function.

However, the crucial property in Eq. (1.171b) can be developed rigorously as the limit of a **sequence** of functions, a distribution. For example, the delta function may be approximated by the

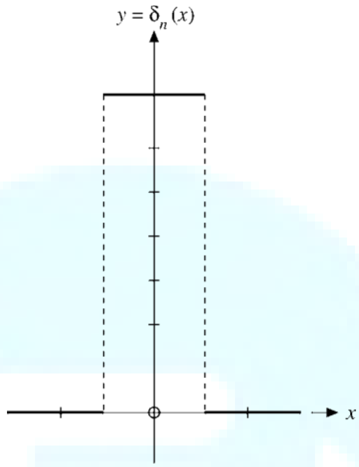

FIGURE 1.37 δ-Sequence function.

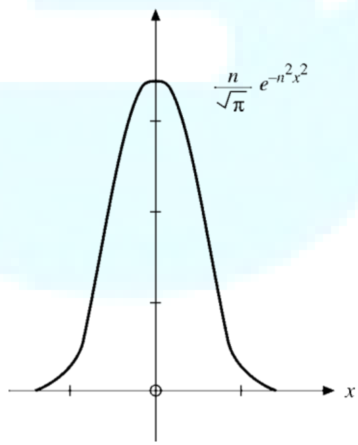

FIGURE 1.38 δ-Sequence function.

---

[1] The delta function is frequently invoked to describe very short-range forces, such as nuclear forces. It also appears in the normalization of continuum wave functions of quantum mechanics. Compare Eq. (1.193c) for plane-wave eigenfunctions.

sequences of functions, Eqs. (1.172) to (1.175) and Figs. 1.37 to 1.40:

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & x > \frac{1}{2n} \end{cases}$$

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} \exp(-n^2x^2)$$

$$\delta_n(x) = \frac{n}{\pi} \cdot \frac{1}{1+n^2x^2}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt}\, dt.$$

(1.172)

(1.173)

(1.174)

(1.175)

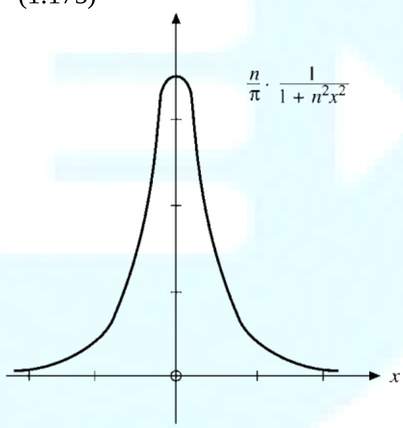

FIGURE 1.39 δ-Sequence function.

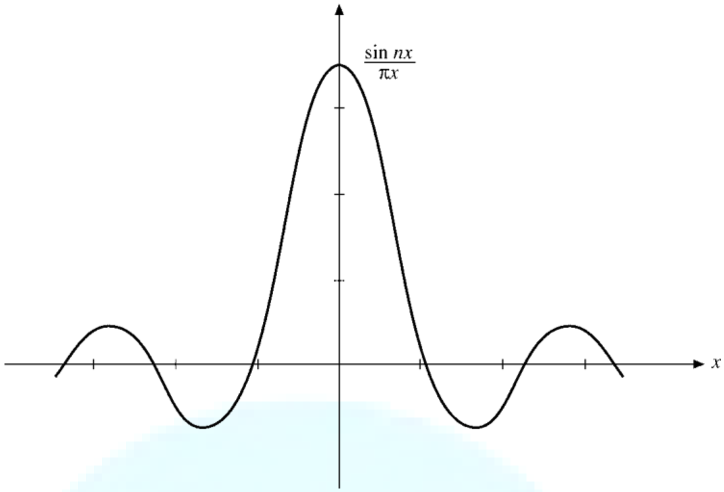

FIGURE 1.40 δ-Sequence function.

These approximations have varying degrees of usefulness. Equation (1.172) is useful in providing a simple derivation of the integral property, Eq. (1.171b). Equation (1.173) is convenient to differentiate. Its derivatives lead to the Hermite polynomials. Equation (1.175) is particularly useful in Fourier analysis and in its applications to quantum mechanics. In the theory of Fourier series, Eq. (1.175) often appears (modified) as the Dirichlet kernel:

$$\delta_n(x) = \frac{1}{2\pi} \frac{\sin[(n+\frac{1}{2})x]}{\sin(\frac{1}{2}x)} . \qquad (1.176)$$

In using these approximations in Eq. (1.171b) and later, we assume that $f(x)$ is well behaved—it offers no problems at large x.

For most physical purposes such approximations are quite adequate. From a mathematical point of view the situation is still unsatisfactory: The limits

$$\lim_{n\to\infty} \delta_n(x)$$

do not exist.

A way out of this difficulty is provided by the theory of distributions. Recognizing that Eq. (1.171b) is the fundamental property, we focus our attention on it rather than on $\delta(x)$ itself. Equations (1.172) to (1.175) with $n = 1,2,3,\ldots$ may be interpreted as **sequences** of normalized functions:

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1 . \qquad (1.177)$$

The sequence of integrals has the limit

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)\,dx = f(0). \tag{1.178}$$

Note that Eq. (1.178) is the limit of a sequence of integrals. Again, the limit of $\delta_n(x)$, $n \to \infty$, does not exist. (The limits for all four forms of $\delta_n(x)$ diverge at $x = 0$.)

We may treat $\delta(x)$ consistently in the form

$$\int_{-\infty}^{\infty}\delta(x)f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)\,dx. \tag{1.179}$$

$\delta(x)$ is labeled a distribution (not a function) defined by the sequences $\delta_n(x)$ as indicated in Eq. (1.179). We might emphasize that the integral on the left-hand side of Eq. (1.179) is not a Riemann integral.[2] It is a limit.

This distribution $\delta(x)$ is only one of an infinity of possible distributions, but it is the one we are interested in because of Eq. (1.171b).

From these sequences of functions we see that Dirac's delta function must be even in x, $\delta(-x) = \delta(x)$.

The integral property, Eq. (1.171b), is useful in cases where the argument of the delta function is a function $g(x)$ with simple zeros on the real axis, which leads to the rules

$$\delta(ax) = \frac{1}{a}\delta(x), \qquad a > 0, \tag{1.180}$$

$$\delta\big(g(x)\big) = \sum_{\substack{a,\\ g(a)=0,\\ g'(a)\neq 0}} \frac{\delta(x-a)}{|g'(a)|}. \tag{1.181a}$$

Equation (1.180) may be written

$$= \frac{1}{|a|}\delta(x) \text{ for } a < 0$$

$$\int_{-\infty}^{\infty} f(x)\delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)\,dy = \frac{1}{a}f(0),$$

applying Eq. (1.171b). Equation (1.180) may be written as $\delta(ax)$. To prove Eq. (1.181a) we decompose the integral

$$\int_{-\infty}^{\infty} f(x)\delta\big(g(x)\big)\,dx = \sum_a \int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta\big((x-a)g'(a)\big)\,dx \tag{1.181b}$$

into a sum of integrals over small intervals containing the zeros of $g(x)$. In these intervals, $g(x) \approx g(a) + (x-a)g'(a) = (x-a)g'(a)$. Using Eq. (1.180) on the right-hand side of Eq. (1.181b) we obtain the integral of Eq. (1.181a).

---

[2] It can be treated as a Stieltjes integral if desired. $\delta(x)dx$ is replaced by $du(x)$, where $u(x)$ is the Heaviside step function (compare Exercise 1.15.13).

Using integration by parts we can also **define the derivative** $\delta'(x)$ of the Dirac delta function by the relation

$$\int_{-\infty}^{\infty} f(x)\delta'(x-x')\,dx = -\int_{-\infty}^{\infty} f'(x)\delta(x-x')\,dx = -f'(x'). \tag{1.182}$$

We use $\delta(x)$ frequently and call it the Dirac delta function[3] —for historical reasons. Remember that it is not really a function. It is essentially a shorthand notation, defined implicitly as the limit of integrals in a sequence, $\delta_n(x)$, according to Eq. (1.179). It should be understood that our Dirac delta function has significance only as part of an integrand. In this spirit, the linear operator $\int dx\,\delta(x-x_0)$ operates on $f(x)$ and yields $f(x_0)$:

$$\mathcal{L}(x_0)f(x) \equiv \int_{-\infty}^{\infty} \delta(x-x_0)f(x)\,dx = f(x_0). \tag{1.183}$$

It may also be classified as a linear mapping or simply as a generalized function. Shifting our singularity to the point $x = x'$, we write the Dirac delta function as $\delta(x - x')$. Equation (1.171b) becomes

$$\int_{-\infty}^{\infty} f(x)\delta(x-x')\,dx = f(x'). \tag{1.184}$$

As a description of a singularity at $x = x'$, the Dirac delta function may be written as $\delta(x - x')$ or as $\delta(x' - x)$. Going to three dimensions and using spherical polar coordinates, we obtain

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \delta(\mathbf{r})r^2\,dr\,\sin\theta\,d\theta\,d\varphi = \iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,dx\,dy\,dz = 1. \tag{1.185}$$

This corresponds to a singularity (or source) at the origin. Again, if our source is at $\mathbf{r} = \mathbf{r}_1$, Eq. (1.185) becomes

$$\iiint \delta(\mathbf{r}_2-\mathbf{r}_1)r_2^2\,dr_2\,\sin\theta_2\,d\theta_2\,d\varphi_2 = 1. \tag{1.186}$$

Example 1.15.1 TOTAL CHARGE INSIDE A SPHERE

Consider the total electric flux $\mathbf{E} \cdot d\sigma$ out of a sphere of radius R around the origin surrounding n charges e_j, located at the points $\mathbf{r}_j$ with $r_j < R$, that is, inside the sphere. The electric field strength $\mathbf{E} = -\nabla\varphi(\mathbf{r})$, where the potential

[3] Dirac introduced the delta function to quantum mechanics. Actually, the delta function can be traced back to Kirchhoff, 1882. For further details see M. Jammer, *The Conceptual Development of Quantum Mechanics*. New York: McGraw–Hill (1966), p. 301.

$$\varphi = \sum_{j=1}^{n} \frac{e_j}{|\mathbf{r} - \mathbf{r}_j|} = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r$$

is the sum of the Coulomb potentials generated by each charge and the total charge density is $\rho(\mathbf{r}) = \sum_j e_j \delta(\mathbf{r} - \mathbf{r}_j)$. The delta function is used here as an abbreviation of a pointlike density. Now we use Gauss' theorem for

$$\oint \mathbf{E} \cdot d\boldsymbol{\sigma} = -\oint \nabla\varphi \cdot d\boldsymbol{\sigma} = -\int \nabla^2\varphi \, d\tau = \int \frac{\rho(\mathbf{r})}{\varepsilon_0} d\tau = \frac{\sum_j e_j}{\varepsilon_0}$$

in conjunction with the differential form of Gauss's law, $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$, and

$$\sum_j e_j \int \delta(\mathbf{r} - \mathbf{r}_j) \, d\tau = \sum_j e_j .$$

Example 1.15.2 PHASE SPACE

In the scattering theory of relativistic particles using Feynman diagrams, we encounter the following integral over energy of the scattered particle (we set the velocity of light $c = 1$):

$$\int d^4p \, \delta(p^2 - m^2) f(p) \equiv \int d^3p \int dp_0 \, \delta(p_0^2 - \mathbf{p}^2 - m^2) f(p)$$

$$= \int_{E>0} \frac{d^3p f(E, \mathbf{p})}{2\sqrt{m^2 + \mathbf{p}^2}} + \int_{E<0} \frac{d^3p f(E, \mathbf{p})}{2\sqrt{m^2 + \mathbf{p}^2}},$$

where we have used Eq. (1.181a) at the zeros $E = \pm\sqrt{m^2 + \mathbf{p}^2}$ of the argument of the delta function. The physical meaning of $\delta(p^2 - m^2)$ is that the particle of mass m and four-momentum $p^\mu = (p_0, \mathbf{p})$ is on its mass shell, because $p^2 = m^2$ is equivalent to $E = \pm\sqrt{m^2 + \mathbf{p}^2}$. Thus, the on-mass-shell volume element in momentum space is the Lorentz invariant $\frac{d^3p}{2E}$, in contrast to the nonrelativistic d^3p of momentum space. The fact that a negative energy occurs is a peculiarity of relativistic kinematics that is related to the antiparticle.

Delta Function Representation by Orthogonal Functions

Dirac's delta function[4] can be expanded in terms of any basis of real orthogonal functions $\{\varphi_n(x), n = 0,1,2,...\}$. Such functions will occur in Chapter 10 as solutions of ordinary differential equations of the Sturm–Liouville form.

They satisfy the orthogonality relations

$$\int_a^b \varphi_m(x)\varphi_n(x)\,dx = \delta_{mn}, \tag{1.187}$$

where the interval (a,b) may be infinite at either end or both ends. [For convenience we assume that φ_n has been defined to include $(w(x))^{1/2}$ if the orthogonality relations contain an additional positive weight function $w(x)$.] We use the φ_n to expand the delta function as

$$\delta(x-t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x), \tag{1.188}$$

where the coefficients a_n are functions of the variable t. Multiplying by $\varphi_m(x)$ and integrating over the orthogonality interval (Eq. (1.187)), we have

$$a_m(t) = \int_a^b \delta(x-t)\varphi_m(x)\,dx = \varphi_m(t) \tag{1.189}$$

or

$$\delta(x-t) = \sum_{n=0}^{\infty} \varphi_n(t)\varphi_n(x) = \delta(t-x). \tag{1.190}$$

This series is assuredly not uniformly convergent (see Chapter 5), but it may be used as part of an integrand in which the ensuing integration will make it convergent (compare Section 5.5).

Suppose we form the integral $\int F(t)\delta(t-x)\,dx$, where it is assumed that $F(t)$ can be expanded in a series of orthogonal functions $\varphi_p(t)$, a property called *completeness*. We then obtain

$$\int F(t)\delta(t-x)\,dt = \int \sum_{p=0}^{\infty} a_p\varphi_p(t) \sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(t)\,dt$$

$$= \sum_{p=0}^{\infty} a_p\varphi_p(x) = F(x), \tag{1.191}$$

[4] This section is optional here. It is not needed until Chapter 10.

the cross products $\int \varphi_p \varphi_n \, dt \, (n \quad p)$ = vanishing by orthogonality (Eq. (1.187)). Referring back to the definition of the Dirac delta function, Eq. (1.171b), we see that our series representation, Eq. (1.190), satisfies the defining property of the Dirac delta function and therefore is a representation of it. This representation of the Dirac delta function is called **closure**. The assumption of completeness of a set of functions for expansion of $\delta(x - t)$ yields the closure relation. The converse, that closure implies completeness, is the topic of Exercise 1.15.16.

Integral Representations for the Delta Function

Integral transforms, such as the Fourier integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt$$

of Chapter 15, lead to the corresponding integral representations of Dirac's delta function. For example, take

$$\delta_n(t - x) = \frac{\sin n(t - x)}{\pi(t - x)} = \frac{1}{2\pi} \int_{-n}^{n} \exp\big(i\omega(t - x)\big) \, d\omega, \tag{1.192}$$

using Eq. (1.175). We have

$$f(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\delta_n(t - x) \, dt, \tag{1.193a}$$

where $\delta_n(t - x)$ is the sequence in Eq. (1.192) defining the distribution $\delta(t - x)$. Note that Eq. (1.193a) assumes that $f(t)$ is continuous at $t = x$. If we substitute Eq. (1.192) into Eq. (1.193a) we obtain

$$f(x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-n}^{n} \exp\big(i\omega(t - x)\big) \, d\omega \, dt. \tag{1.193b}$$

Interchanging the order of integration and then taking the limit as $n \to \infty$, we have the Fourier integral theorem, Eq. (15.20).

With the understanding that it belongs under an integral sign, as in Eq. (1.193a), the identification

$$\boxed{\delta(t - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\big(i\omega(t - x)\big) \, d\omega} \tag{1.193c}$$

provides a very useful integral representation of the delta function.

When the Laplace transform (see Sections 15.1 and 15.9)

$$L_\delta(s) = \int_0^{\infty} \exp(-st)\delta(t - t_0) = \exp(-st_0), \qquad t_0 > 0 \tag{1.194}$$

is inverted, we obtain the complex representation

$$\delta(t-t_0)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\bigl(s(t-t_0)\bigr)\,ds, \tag{1.195}$$

which is essentially equivalent to the previous Fourier representation of Dirac's delta function.

Exercises

1.15.1 Let

$$\delta_n(x)=\begin{cases}0, & x<-\frac{1}{2n},\\ n, & -\frac{1}{2n}<x<\frac{1}{2n}\\ 0, & \frac{1}{2n}<x.\end{cases},$$

Show that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}f(x)\delta_n(x)\,dx=f(0),$$

assuming that $f(x)$ is continuous at $x = 0$.

1.15.2 Verify that the sequence $\delta_n(x)$, based on the function

$$\delta_n(x)=\begin{cases}0, & x<0,\\ ne^{-nx}, & x>0\end{cases},$$

is a delta sequence (satisfying Eq. (1.178)). Note that the singularity is at +0, the positive side of the origin.

Hint. Replace the upper limit (∞) by c/n, where c is large but finite, and use the mean value theorem of integral calculus.

1.15.3 For

$$\delta_n(x)=\frac{n}{\pi}\cdot\frac{1}{1+n^2x^2},$$

(Eq. (1.174)), show that

$$\int_{-\infty}^{\infty}\delta_n(x)\,dx=1.$$

1.15.4 Demonstrate that $\delta_n = \sin nx/\pi x$ is a delta distribution by showing that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}f(x)\frac{\sin nx}{\pi x}dx=f(0).$$

Assume that $f(x)$ is continuous at $x = 0$ and vanishes as $x \to \pm\infty$. *Hint*. Replace x by y/n and take $\lim n \to \infty$ **before** integrating.

1.15.5 Fejer's method of summing series is associated with the function

$$\delta_n(t) = \frac{1}{2\pi n}\left[\frac{\sin(nt/2)}{\sin(t/2)}\right]^2.$$

Show that $\delta_n(t)$ is a delta distribution, in the sense that

$$\lim_{n\to\infty} \frac{1}{2\pi n}\int_{-\infty}^{\infty} f(t)\left[\frac{\sin(nt/2)}{\sin(t/2)}\right]^2 dt = f(0).$$

1.15.6 Prove that

$$\delta\left[a(x-x_1)\right] = \frac{1}{a}\delta(x-x_1).$$

Note. If $\delta[a(x-x_1)]$ is considered even, relative to x_1, the relation holds for negative a and $1/a$ may be replaced by $1/|a|$.

1.15.7 Show that

$$\delta\left[(x-x_1)(x-x_2)\right] = \left[\delta(x-x_1)+\delta(x-x_2)\right]/|x_1-x_2|.$$

Hint. Try using Exercise 1.15.6.

1.15.8 Using the Gauss error curve delta sequence ($\delta_n = \frac{n}{\sqrt{\pi}}e^{-n^2x^2}$), show that

$$x\frac{d}{dx}\delta(x) = -\delta(x),$$

treating $\delta(x)$ and its derivative as in Eq. (1.179).

1.15.9 Show that

$$\int_{-\infty}^{\infty}\delta'(x)f(x)\,dx = -f'(0).$$

Here we assume that $f'(x)$ is continuous at $x = 0$.

1.15.10 Prove that

$$\delta\big(f(x)\big) = \left|\frac{df(x)}{dx}\right|_{x=x_0}^{-1}\delta(x-x_0),$$

where x_0 is chosen so that $f(x_0) = 0$.

Hint. Note that $\delta(f)df = \delta(x)\,dx$.

1.15.11 Show that in spherical polar coordinates $(r, \cos\theta, \varphi)$ the delta function $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ becomes

$$\frac{1}{r_1^2}\delta(r_1-r_2)\delta(\cos\theta_1-\cos\theta_2)\delta(\varphi_1-\varphi_2).$$

Generalize this to the curvilinear coordinates (q_1, q_2, q_3) of Section 2.1 with scale factors h_1, h_2, and h_3.

1.15.12 A rigorous development of Fourier transforms[5] includes as a theorem the relations

[5] I. N. Sneddon, *Fourier Transforms*. New York: McGraw-Hill (1951).

$$\lim_{a\to\infty}\frac{2}{\pi}\int_{x_1}^{x_2} f(u+x)\frac{\sin ax}{x}dx$$

$$=\begin{cases} f(u+0)+f(u-0), & x_1<0<x_2 \\ f(u+0), & x_1=0<x_2 \\ f(u-0), & x_1<0=x_2 \\ 0, & x_1<x_2<0 \text{ or } 0<x_1<x_2. \end{cases}$$

Verify these results using the Dirac delta function.

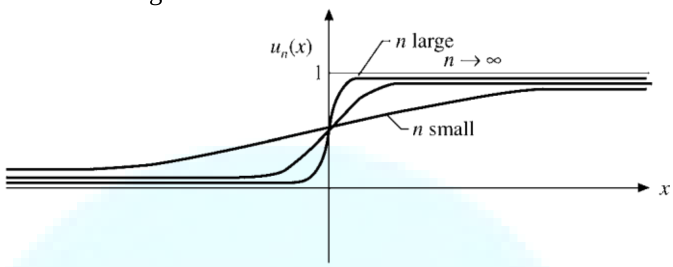

FIGURE 1.41 $\frac{1}{2}[1+\tanh nx]$ and the Heaviside unit step function.

1.15.13 (a) If we define a sequence $\delta_n(x) = n/(2\cosh^2 nx)$, show that

$$\int_{-\infty}^{\infty}\delta_n(x)\,dx = 1,$$ independent of n.

(b) Continuing this analysis, show that[6]

$$\int_{-\infty}^{x}\delta_n(x)\,dx = \frac{1}{2}[1+\tanh nx] \equiv u_n$$

$$\lim_{n\to\infty} u_n(x) = \begin{cases} 0, & x<0, \\ 1, & x>0. \end{cases} \quad (x),$$

This is the Heaviside unit step function (Fig. 1.41).

1.15.14 Show that the unit step function $u(x)$ may be represented by

$$u(x) = \frac{1}{2} + \frac{1}{2\pi i}P\int_{-\infty}^{\infty} e^{ixt}\frac{dt}{t},$$

where P means Cauchy principal value (Section 7.1).

1.15.15 As a variation of Eq. (1.175), take

[6] Many other symbols are used for this function. This is the AMS-55 (see footnote 4 on p. 330 for the reference) notation: u for unit.

$$\delta_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt-|t|/n}\,dt.$$

Show that this reduces to $(n/\pi)1/(1 + n^2x^2)$, Eq. (1.174), and that

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1 .$$

Note. In terms of integral transforms, the initial equation here may be interpreted as either a Fourier exponential transform of $e^{-|t|/n}$ or a Laplace transform of e^{ixt}.

1.15.16 (a) The Dirac delta function representation given by Eq. (1.190),

$$\delta(x - t) = \sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(t),$$

is often called the **closure relation**. For an orthonormal set of real functions, φ_n, show that closure implies completeness, that is, Eq. (1.191) follows from Eq. (1.190).

Hint. One can take

$$F(x) = \int F(t)\delta(x - t)\,dt.$$

(b) Following the hint of part (a) you encounter the integral $\int F(t)\varphi_n(t)dt$. How do you know that this integral is finite?

1.15.17 For the finite interval $(-\pi,\pi)$ write the Dirac delta function $\delta(x - t)$ as a series of sines and cosines: $\sin nx, \cos nx, n = 0,1,2,\ldots$. Note that although these functions are orthogonal, they are not normalized to unity.

1.15.18 In the interval $(-\pi, \pi)$, $\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2)$.

(a) Write $\delta_n(x)$ as a Fourier cosine series.

(b) Show that your Fourier series agrees with a Fourier expansion of $\delta(x)$ in the limit as $n \to \infty$.

(c) Confirm the delta function nature of your Fourier series by showing that for any $f(x)$ that is finite in the interval $[-\pi,\pi]$ and continuous at $x = 0$,

$$\int_{-\pi}^{\pi} f(x)\left[\text{Fourier expansion of } \delta_\infty(x)\right]dx = f(0).$$

1.15.19 (a) Write $\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2)$ in the interval $(-\infty,\infty)$ as a Fourier integral and compare the limit $n \to \infty$ with Eq. (1.193c).

(b) Write $\delta_n(x) = n\exp(-nx)$ as a Laplace transform and compare the limit $n \to \infty$ with Eq. (1.195).

Hint. See Eqs. (15.22) and (15.23) for (a) and Eq. (15.212) for (b).

1.15.20 (a) Show that the Dirac delta function $\delta(x - a)$, expanded in a Fourier sine series in the half-interval $(0,L)$, $(0 < a < L)$, is given by

$$\delta(x-a) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Note that this series actually describes

$$-\delta(x + a) + \delta(x - a) \qquad \text{in the interval } (-L,L).$$

(b) By integrating both sides of the preceding equation from 0 to x, show that the cosine expansion of the square wave

$$f(x) = \begin{cases} 0, & 0 \leqslant x < a \\ 1, & a < x < L, \end{cases}$$

ENTRI

is, for 0 x < L,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

(c) Verify that the term

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right) \quad \text{is} \quad \langle f(x)\rangle \equiv \frac{1}{L}\int_0^L f(x)\,dx.$$

1.15.21 Verify the Fourier cosine expansion of the square wave, Exercise 1.15.20(b), by direct calculation of the Fourier coefficients. **1.15.22** We may define a sequence

$$\delta_n(x) \quad n, \quad = \begin{cases} n, & |x| < 1/2 \\ 0, & x > 1/2n. \end{cases}$$

| |

(This is Eq. (1.172).) Express $\delta_n(x)$ as a Fourier integral (via the Fourier integral theorem, inverse transform, etc.). Finally, show that we may write

$$= \lim_{n\to\infty}\delta_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dk. \quad \delta(x)$$

1.15.23 Using the sequence

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2),$$

show that

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dk. \quad \delta(x)$$

Note. Remember that $\delta(x)$ is defined in terms of its behavior as part of an integrand—especially Eqs. (1.178) and (1.189).

1.15.24 Derive sine and cosine representations of $\delta(t - x)$ that are comparable to the exponential representation, Eq. (1.193c).

ANS. $\frac{2}{\pi}\int_0^{\infty}\sin\omega t\sin\omega x\,d\omega,\ \frac{2}{\pi}\int_0^{\infty}\cos\omega t\cos\omega x\,d\omega.$

1.16 HELMHOLTZ'S THEOREM

In Section 1.13 it was emphasized that the choice of a magnetic vector potential **A** was not unique. The divergence of **A** was still undetermined. In this section two theorems about the divergence and curl of a vector are developed. The first theorem is as follows:

A vector is uniquely specified by giving its divergence and its curl within a simply connected region (without holes) and its normal component over the boundary.