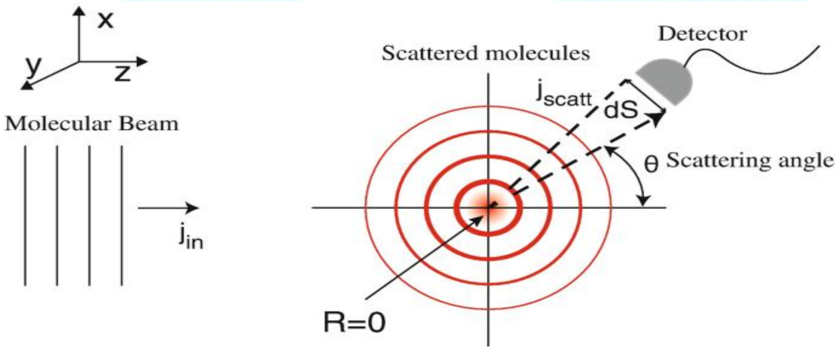
Elementary theory of Scattering

Scattering Theory

- Much of our understanding about the structure of matter is extracted from the scattering of particles.
- It is through scattering experiment that important building blocks of matter, such as atomic nucleus, the nucleons, and the various quarks, have been discovered.

Scattering and Cross Section

- In a scattering experiment, one observes the collisions between a beam of incident particles and a target material.
- The total number of collisions over the duration of the experiment is proportional to the total number of incident particles and to the number of target particles per unit area in the path of the beam.
- After scattering , those particles that do not interact with the target continue their motion (undisturbed) in the forward direction, but those that interact with the target get scattered (deflected) at some angle.



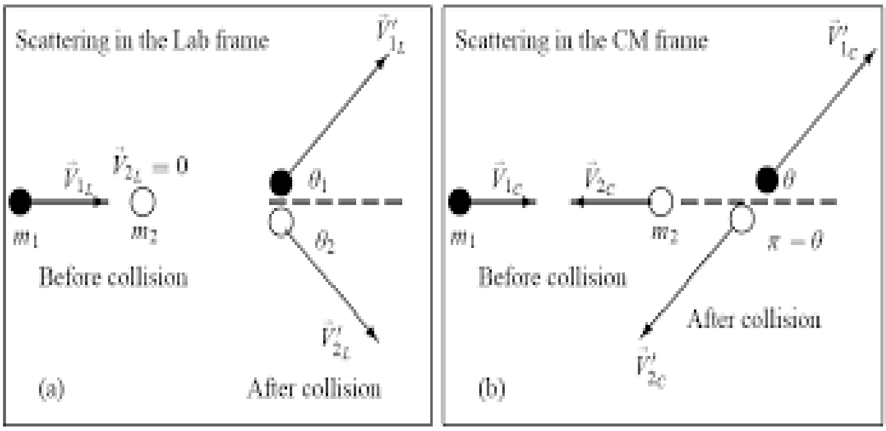
- The number of particle coming out varies from one direction to the other. The number of particles scattered into an element of solid angle $d\Omega$ ($d\Omega = \sin\theta d\theta\, d\varphi$) Is proportional to a quantity that plays a central role in physical scattering : th differential cross section. The differential cross section, which is denoted by $d\sigma(\theta, \emptyset)/d\Omega$, is defined as the number of particles scattered into an element of solid angle of $d\Omega$ in the direction $(\theta,\emptyset)$ per unit time and incident flux.

$$d\sigma(\theta, \emptyset)/d\Omega, = \frac{1}{J_{inc}} \frac{d\, N(\theta,\emptyset)}{d\Omega}$$

- Where J_{inc} is the incident flux (or the incident current density) ; it is equal to the number of incident particles per area per unit time. We can verify that $d\sigma / d\Omega$ has the dimensions of an area ; hence it is appropriate to call it a differential cross section.
- The relationship between $d\sigma / d\Omega$ and the total cross section σ is

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \frac{d\sigma(\theta,\varphi)}{d\Omega} d\varphi$$

- Most scattering experiments are carried out in the laboratory frame in which target is initially at rest while the projectiles are moving.
- Calculations of the cross sections are generally easier to perform within the centre of mass (CM) frame in which the centre of mass of the projectiles - target system is rest (before and after collision)
- In order to be able to compare the experimental measurements with the theoretical calculations, one has to know how to transform the cross sections from one frame into the other.
- The total cross is the same in both the frames, since the total number of collisions that take place does not depend on the frame in which the observation is carried out.
- As far the differential cross section $d\sigma(\theta, \emptyset)/d\Omega$, they are not the same in both frames, since the scattering angles (θ,∅) are frame dependent.



$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

Which, using $\cos\theta_1 = 1/\sqrt{tan^2 1 + 1}$, becomes

$$\cos\theta_1 = \frac{\cos\theta + \frac{m_1}{m_2}}{\sqrt{1 + \frac{m1^2}{m2^2 +} + 2\frac{m_1}{m_2}\cos\theta}}$$

$$\tan\theta = \frac{Sin\,\theta}{-\cos\theta + V_{CM/V_{2c}'}} = \frac{\sin\theta}{1-\cos\theta} = \cot\left(\frac{\theta}{2}\right)$$

From this $\theta_2 = \dfrac{\pi - \theta}{2}$

$$\left(\frac{d\sigma}{d'\Omega_1}\right)_{Lab} = \frac{\left(1 + \frac{m1^2}{m2^2 +} + 2\frac{m_1}{m_2}\cos\theta\right)^{3/2}}{1 + \frac{m_1}{m_2}\cos\theta}\left(\frac{d\sigma}{d'\Omega}\right)_{CM}$$

$$\left(\frac{d\sigma}{d'\Omega_2}\right)_{Lab} = 4\cos\theta_2\left(\frac{d\sigma}{d'\Omega_2}\right)_{Lab} = 4\sin\left(\frac{\theta}{2}\right)\left(\frac{d\sigma}{d'\Omega_1}\right)_{cm}$$

Limiting case ; (a) if $m_1 >> m_2$, or when $\dfrac{m_1}{m_2} \longrightarrow 0,$ the lab and CM results be the same.

$$\theta_1 = \theta \text{ and } \left(\frac{d\sigma}{d'\Omega_1}\right)_{Lab} = \left(\frac{d\sigma}{d'\Omega_1}\right)_{cm}$$

(b) if $m_1 = m_2$ then

$\theta_1 = \theta/2$

$$\left(\frac{d\sigma}{d'\Omega_1}\right)_{Lab} = 4\left(\frac{d\sigma}{d'\Omega_1}\right)_{cm}\cos(\theta/2)$$

Scattering Amplitude of Spinless Particles

We consider the case of scattering between two spinless, nonrelativistic particles of masses m_1 and m_2. During the scattering process, the particles interact with one another. If the interaction is time independent, we can describe the two - particle system with stationary states.

$$\Psi(\vec{r_1}, \vec{r_2}, t) = \Psi(\vec{r_1}, \vec{r_2})\, e^{-iE_T t/\hbar}$$

Where E_T is the total energy and $\Psi(\vec{r_1}, \vec{r_2})$, is a solution of the time - independent Schrodinger equation.

$$[-\frac{\hbar^2}{2m_1}\vec{\nabla}_1^{\ 2} - \frac{\hbar^2}{2m_2}\vec{\nabla}_2^{\ 2} + \hat{V}\,(\vec{r_1}\,,\vec{r_2})\,]\,\Psi\,(\vec{r_1}\,,\vec{r_2}) = E_T\,\Psi\,(\vec{r_1}\,,\vec{r_2}),$$

$\hat{V}\,(\vec{r_1}\,,\vec{r_2})$ is the potential representing the interaction between two particles. We can reduce the eigenvalue problem to two decoupled eigenvalue problems; one for centre of mass (CM). which moves like a free particle with reduced mass $\mu = m_1 m_2/\,(m_1+m_2\,)$ which moves in the potential $\hat{V}(\,r)$.

$$\frac{-\hbar^2}{2\mu}\vec{\nabla}^{\,2}{}_1\,\Psi(\vec{r}) + \hat{V}(\,r)\,\Psi(\,r\,) = E\,\Psi(\vec{r}\,)$$

The problem of scattering between two particles is thus reduced to solving this equation.

$$\frac{-\hbar^2}{2\mu}\vec{\nabla}^{\,2}\,\Psi(\overrightarrow{r}) + \hat{V}(\,r)\,\Psi(\,r\,) \ = \ E\,\Psi(\vec{r}\,)$$

$$(\,\nabla^{\,2} + k_0^{\ 2}\,)\,\emptyset_{inc}\,(\,\vec{r}\,) = 0$$

Where $k_0^{\ 2} = 2\mu E/\hbar^2$. In this case μ behaves as a free particle before collision nd hence can be described by a plane wave.

$$\emptyset_{inc}\,(\,\vec{r}\,) = A\,e^{ikz}$$

Where k_0 is the wave vector associated with the incident particle and A is normalization factor. Thus, prior to the interaction with the target, the particles of incident beam are independent of each other , they move like free particles, each with momentum $\vec{p} = \hbar k_0$

When the incident wave collides or interacts with the target, an outgoing wave $\emptyset_{ze}\,(\vec{r})$ is scattered out. In the case of an isotropic scattering, the scattered wave is spherically symmetric, having the form e^{ikr}/r . In general, however the scattered wave is not spherically symmetric ; its amplitude depends on the direction (θ,Ø) along which it is detected and hence

$$\emptyset_{ze}\,(\vec{r}) = A\,f\,\,(\theta,\emptyset)\,\frac{e^{ikr}}{r}$$

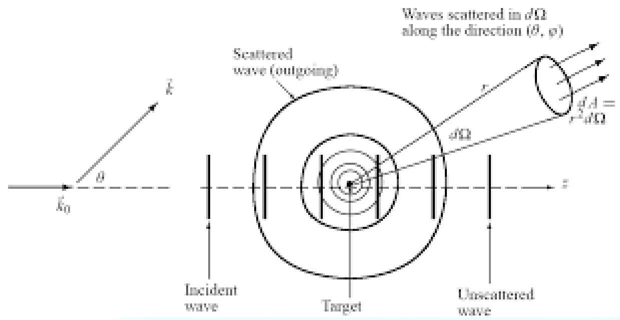
Where f (θ, Ø) is called scattering amplitude , $\vec{k}$ is the wave vector associated with the scattered particle , and θ is the angle between $\overrightarrow{k_0}$ and $\vec{k}$.

After the scattering has taken place , the total wave consists of a superposition of the incident plane wave and the scattered wave.

$$\Psi(\vec{r}) = \phi_{inc}(\vec{r}) + \phi_{ze}(\vec{r})$$

$$\Psi(\vec{r}) = A\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r} \right]$$

Where A is a normalization factor ; since A has no effect on the cross section.we will take it equal to one.



We shall next deduce the relation between the scattering amplitude f(θ) and the differential scattering cross - section σ (θ). The probability current density vector j (r,t) is given by

$$J(r,t) = \frac{-\hbar^2}{2\mu}\left[\Psi^*\nabla\Psi - (\nabla\Psi)^*\Psi \right]$$

If j(r,t) is calculated with the wave function, we get interference terms between the incident and scattered waves. These do not appear in the usual experimental arrangements and therefor we can calculate the incident and scattered probability current densities by substituting the two parts of Ψ separately.

$$j_i = \frac{\hbar k}{\mu}|A|^2 = \frac{P|A|^2}{\mu} = \mathrm{v}\,|A|^2$$

$$j_s = \frac{\hbar}{2i\mu}|A|^2\,|f(\theta)|^2\,\frac{2ik}{r^2} = \mathrm{v}\,|A|^2\,|f(\theta)|^2\,\frac{1}{r^2}$$

Where $1/r^2$ is the solid angle subtended by unit area of the detector at the scattering centre. Also

$$\sigma(\theta) = \frac{\mathrm{v}\,|A|^2\,|f(\theta)|^2}{\mathrm{v}\,|A|^2} = |f(\theta)|^2$$

The scattering amplitude is thus related to the experimentally observable differential scattering cross - section. Since σ (θ) has the dimension of $(length)^2$

f (θ) has the dimension of length.

The Born Approximation

Born used an iteractive procedure for the evaluation of $\Psi(r')$. In the first Bron - approximation $\Psi(r')$ in the integral is replaced by the incoming plane wave exp (ik . r') . This leads to an improved value for the wave function Ψ(r) which is used in the second Born approximation .

Replacing $\Psi(r')$ by exp (ik . r')

$$f(\theta) = -\frac{1}{4\pi}\int exp\,(i(k - k').r')\,U(r)'\,d\tau'$$

Where K and k' are the wave vectors in the incident and scattered directions, respectively. The quantity $(k - k')\hbar = q\hbar$ is then the momentum transfer from the incident particle to the scattering potential . Change in momentum qħ due to collision is given by

$$q\hbar = (k - k')\hbar \text{ or } |q| = 2|k|\sin\frac{\theta}{2}$$

Replacing $(k - k')$ by q in the equ of f(θ)

$$f(\theta) = -\frac{1}{4\pi}\int exp\,(iq\,.r')\,U(r)'\,d\tau'$$

The angular integration of above equ can easily carried out by taking the direction of q as the polar axis. Denoting the angle between q and r' by θ'. Also the integration over Ø gives 2π. The θ - integral can easily be evaluated writing

$$\cos\theta' = x \text{ or } -\sin\theta'\,d\theta' = dx$$

Again simplifying the equation we get

$$f(\theta) = -\frac{2\mu}{\hbar^2}\int_0^\infty \frac{sin\,(qr')}{qr'}\,V(r')r'^2\,dr'$$

Scattering cross section depends on the momentum of the incident particle kħ and scattering angle θ through the combination q = 2k sin (θ/2)