Hermite Polynomials

22.1 Hermite Polynomials

22.1.1 Rodrigues Formula

(See Figure 22.1.)

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n e^{-t^2}}{dt^n}, \quad n = 0, 1, 2, \ldots, \quad -\infty < t < \infty$$

The first few Hermite polynomials are:

$H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, $H_3(t) = 8t^3 - 12t$, $H_4(t) = 16t^4 - 48t^2 + 12$, $H_5(t) = 32t^5 - 160t^3 + 120t$, etc.

22.1.2 Expansion Formula

$$H_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2t)^{n-2k}, \quad [n/2] = \text{largest integer} \le n/2$$

22.1.3 Generating Function

$$w(t, x) = e^{2tx - x^2} = \sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n w}{\partial x^n} \right]_{x=0} x^n, \quad |x| < \infty$$

$n!$ $n!$

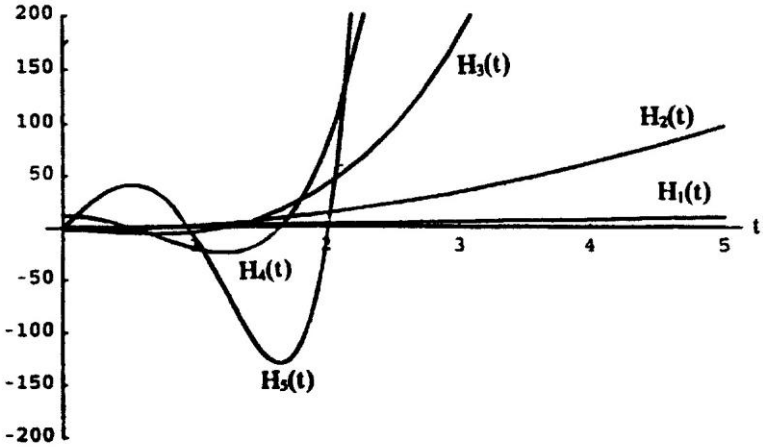

FIGURE 22.1

22.1.4 Even and Odd n

$$H_n(-t) = (-1)^n H_n(t)\,;\quad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!},\quad H_{2n+1}(0) = 0$$

22.2 Recurrence Relation

22.2.1 Recurrence Relation

1. $H_{n+1}(t) - 2t\,H_n(t) + 2nH_{n-1}(t)$ $\quad n = 1, 2, \cdots$
2. $H_n'(t) = 2nH_{n-1}(t)$ $\quad n = 1, 2, \cdots$

   $H_{n+1}'(t) = 2(n+1)H_n(t)$ $\quad n = 1, 2, \cdots$
3. $H_{n+1}(t) - 2t\,H_n(t) + H_n'(t) = 0$ $\quad n = 0, 1, 2, \cdots$

22.2.2 Hermite Differential Equation

$$H_n''(t) - 2t\,H_n'(t) + 2nH_n(t) = 0 \qquad n = 0, 1, 2\cdots$$

which implies that the Hermite polynomials are the solution of the second-order ordinary differential equation.

22.3 Integral Representation

$$H_n(t) = \frac{(-j)^n 2^n e^{t^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+2jtx} x^n \, dx \qquad n = 0, 1, 2, \cdots$$

$$e^{-t^2/2} H_n(t) = \frac{1}{j^n \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{jty} e^{-y^2/2} H_n(y) \, dy \qquad n = 0, 1, 2, \cdots$$

$$e^{-t^2/2} H_{2m}(t) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m}(y) \cos ty \, dy$$

$$e^{-t^2/2} H_{2m+1}(t) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m+1}(y) \sin ty \, dy \qquad m = 0, 1, 2, \cdots$$

22.4 Hermite Series

22.4.1 Orthogonality Property

$$\int_{-\infty}^{\infty} e^{-t^2} H_m(t) H_n(t) \, dt = 0 \quad \text{if} \quad m \neq n$$

and

$$\int_{-\infty}^{\infty} e^{-t^2} H_n^2(t) \, dt = 2^n n! \sqrt{\pi} \qquad n = 0, 1, 2, \cdots$$

22.4.2 Orthonormal Hermite Polynomials

$$\phi_n(t) = (2^n n! \sqrt{\pi})^{-1/2} e^{-t^2/2} H_n(t) \qquad n = 0, 1, 2, \cdots, \; -\infty < t < \infty$$

22.4.3 Hermite Series

$$f(t) = \sum_{n=0}^{\infty} C_n H_n(t) \qquad -\infty < t < \infty$$

$$C_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(t) H_n(t)\,dt \qquad n = 0, 1, 2, \cdots$$

where $f(t)$ is piecewise smooth in every finite interval and $\int_{-\infty}^{\infty} e^{-t^2} f^2(t)\,dt < \infty$.

Example

$$f(t) = t^4 = \sum_{n=0}^{\infty} C_{2n} H_{2n}(t)\text{, since } f(t) \text{ is even.}$$

$$C_{2n} = \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} t^4 H_{2n}(t)\,dt$$

$$= \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \int_{-\infty}^{\infty} t^4 \frac{d^{2n}}{dt^{2n}}(e^{-t^2})\,dt = \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \frac{(4)!}{(4-2n)!} \int_{-\infty}^{\infty} e^{-t^2} t^{4-2n}\,dt$$

$$= \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \frac{(4)!}{(4-2n)!} \Gamma\left(4 - n + \frac{1}{2}\right)$$

22.5 Properties of the Hermite Polynomials

TABLE 22.1 Properties of the Hermite Polynomial

1. $H_n(t) = (-1)^n e^{t^2} \frac{d^n e^{-t^2}}{dt^n} \qquad n = 0, 1, 2\cdots$

$[n/2]$ k

$(-1)^k n!$

2. $H_n(t) = \sum_{k=0} \dfrac{(-1)^k n!}{k!(n-2k)!}(2t)^{n-2k}$ $\quad [n/2]$ = largest integer $\le n/2$

3. $e^{2tx-x^2} = \sum_{n=0}^{\infty} H_n(t)\dfrac{x^n}{n!}$

4. $H_{2n}(0) = (-1)^n \dfrac{(2n)!}{n!}$

5. $H_{2n+1}(0) = 0, \quad H_{2n}'(0) = 0, \quad H_{2n+1}'(0) = (-1)^n \dfrac{(2n+2)!}{(n+1)!}$

6. $H_n(-t) = (-1)^n H_n(t)$

7. $H_{2n}(t)$ = even functions, $\quad H_{2n+1}(t)$ = odd functions

12. $H_n(t) = \dfrac{(-j)^n 2^n e^{t^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+j2tx} x^n \, dx$ $\quad n = 0,1,2\cdots$

13. $e^{-t^2/2} H_n(t) = \dfrac{1}{j^n \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{jty} e^{-y^2/2} H_n(y)\, dy$ = integral equation

14. $e^{-t^2/2} H_{2m}(t) = (-1)^m \sqrt{\dfrac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m}(y) \cos ty\, dy$

8. $H_{n+1}(t) - 2tH_n(t) + 2nH_{n-1}(t) = 0$ $\quad n = 1,2,\cdots$

9. $H_n'(t) = 2nH_{n-1}(t)$ $\quad n = 1,2,\cdots$

10. $H_{n+1}(t) - 2tH_n(t) + H_n'(t) = 0$ $\quad n = 0,1,2\cdots$

11. $H_n''(t) - 2tH_n'(t) + 2nH_n(t) = 0$ $\quad n = 0,1,2\cdots$

15. $e^{-t^2/2} H_{2m+1}(t) = (-1)^m \sqrt{\dfrac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m+1}(y) \sin ty\, dy$

16. $\int_{-\infty}^{\infty} e^{-t^2} H_m(t) H_n(t)\, dt = 0$ $\quad$ if $\quad m \ne n$

17. $\int_{-\infty}^{\infty} e^{-t^2} H_n^2(t)\, dt = 2^n n! \sqrt{\pi}$ $\quad n = 0,1,2\cdots$

18. $f(t) = \sum_{n=0}^{\infty} C_n H_n(t) \qquad -\infty < t < \infty$

$$C_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(t) H_n(t)\, dt$$

19. $\int_{-\infty}^{\infty} t^k e^{-t^2} H_n(t)\, dt = 0 \qquad k = 0, 1, \cdots, n-1$

TABLE 22.1 Properties of the Hermite Polynomial (continued)

20. $\int_{-\infty}^{\infty} t^2 e^{-t^2} H_n^2(t)\, dt = \sqrt{\pi}\, 2^n n! \left(n + \frac{1}{2}\right)$

21. $\int_{0}^{\infty} x^n e^{-x^2} H_n(tx)\, dx = \frac{\sqrt{\pi}\, n!}{2} P_n(t)$

22. $\int_{-\infty}^{\infty} e^{-2t^2} H_n^2(t)\, dt = 2^{n-\frac{1}{2}} \Gamma\left(n + \frac{1}{2}\right)$

23. $\frac{d^m H_n(t)}{dt^m} = \frac{2^m n!}{(n-m)!} H_{n-m}(t) \qquad m < n$

24. $\int_{-\infty}^{\infty} e^{-a^2 t^2} H_{2n}(t)\, dt = \frac{(2n)!}{n!} \frac{\sqrt{\pi}}{a} \left(\frac{1-a^2}{a^2}\right)^n \qquad a > 0$

25. $\int_{-\infty}^{\infty} e^{-t^2 + 2bt} H_n(t)\, dt = \sqrt{\pi}\, (2b)^n e^{b^2} \qquad n = 0, 1, 2 \cdots$