Klein - Gordon Equation

The nonrelativistic Schrodinger equation was obtained by replacing p by -iħ∇ and E by $i\hbar\frac{\partial}{\partial t}$ in the classical energy expression of a free particle E $= p^2/(2m)$ and allowing the resulting operator equation to operate on he wave function. The corresponding relativistic energy relation is

$$E^2 = c^2p^2 + m^2c^4$$

Where m is the rest mass of the particle. For convenience rest mass be denoted by m . replacing E and p by the respective operators, we get the operator equation.

$$\hbar^2 \frac{\partial^2}{\partial t^2} = -c^2\hbar^2\nabla^2 + m^2c^4$$

Allowing this operator equation to operate on the wave function

$$-\hbar^2 \frac{\partial^2\Psi(r,t)}{\partial t^2} = -c^2\hbar^2\nabla^2\Psi(r,t) + m^2c^4\Psi(r,t)$$

Which is the Klein - Gordon equation or schrodinger's relativistic equation. Rearranging , we get

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(r,t) = \frac{m^2c^4}{\hbar^2}\Psi(r,t) \quad \text{or}$$

$$\Box\Psi(r,t) = \frac{m^2c^4}{\hbar^2}\Psi(r,t)$$

$$\Box = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$$

Where □ is the de Alembertian operator which is relativistically invariant.

Plane wave solution

The plane wave represented by

$$\Psi(r,t) = \exp[i(k \cdot r - \omega t)]$$

Is an eigenfunction of both energy and momentum operators with eigenvalues ħω and kħ respectively. Substituting the above solution in K-G equation we get

$$(\hbar\omega)^2 = c^2\hbar^2k^2 + m^2c^4$$

This means that the energy eigenvalue can be both positive and negative values. Klein and Gordon were not able to give explanation for negative energy.