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Laguerre Polynomials

23.1 Laguerre Polynomials

23.1.1 Definition

$$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n!\, t^k}{(k!)^2 (n-k)!} \qquad n = 0, 1, 2, \cdots, \quad 0 \le t < \infty$$

Figure 23.1 shows several Laguerre polynomials.

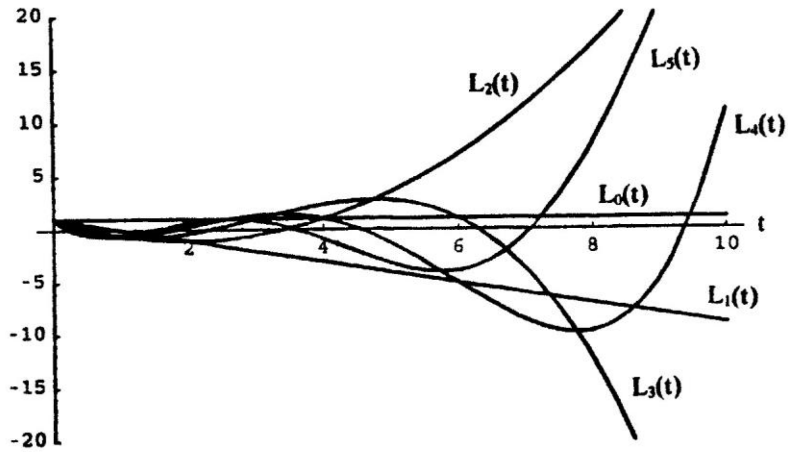

FIGURE 23.1

23.1.2 Rodrigues Formula

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right) \quad n = 0, 1, 2, \cdots$$

23.1.3 Laguerre Polynomials

$$L_0(t) = 1, \quad L_1(t) = -t + 1, \quad L_2(t) = \frac{1}{2!}(t^2 - 4t + 2),$$

$L_3(t) =$

23.1.4 Generating Function

$$w(t,x) = (1-x)^{-1}\exp\left(-\frac{tx}{1-x}\right) = \sum_{n=0}^{\infty} L_n(t)\,x^n$$

$|x| < 1,\ 0 \le t < \infty$

23.2 Recurrence Relations

23.2.1 Recurrence Relations

1. $L_n'(t) - L_{n-1}'(t) + L_{n-1}(t) = 0$ $\quad n = 1, 2, \cdots$
2. $L_{n+1}'(t) = L_n'(t) - L_n(t)$ $\quad n = 1, 2, \cdots$
3. $L_{n-1}'(t) = L_n'(t) + L_{n-1}(t)$ $\quad n = 1, 2, \cdots$
4. $(n+1)L_{n+1}'(t) + (t-1-2n)L_n'(t) + L_n(t) + nL_{n-1}'(t) = 0$ $\quad n = 1, 2, \cdots$
5. $tL_n'(t) = nL_n(t) - nL_{n-1}(t)$ $\quad n = 1, 2, \cdots$

$\frac{1}{3!}(-t^3 + 9t^2 - 18t + 6),\ L_4(t) = \frac{1}{4!}(t^4 - 16t^3 + 72t^2 - 96t + 24)$

23.2.2 Laguerre Equation

$$tL_n''(t) + (1-t)L_n'(t) + nL_n(t) = 0$$

23.3 Laguerre Series

23.3.1 Orthogonality Relation

$$\int_0^{\infty} e^{-t} L_n(t) L_m(t)\,dt = 0 \quad n \ne m$$

23.3.2 Orthonormal Functions

$$\phi_n(t) = e^{-t/2} L_n(t) \quad n = 0, 1, 2, \cdots$$

23.3.3 Laguerre Series

$$f(t) = \sum_{n=0}^{\infty} C_n L_n(t) \quad 0 \le t < \infty$$

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$C = \quad e^{-t} f\, t\, L\, t\, dt(\,) \qquad (\,)\; n = 0\; 1\; 2,\,,$

…

$_n \int_0 \qquad _n$

23.4 Associated Laguerre Polynomials (or Generalized)

23.4.1 Definitions

For a real $a > -1$ the general Laguerre polynomials are defined by the formula

$$L\, t_{a n}(\,) \quad \frac{t_{-a}\, d_n}{\quad} \quad {}_t \quad {}_{-\,+t\,n\,a} \quad = 0$$

1 2, , ⋯

$$= e \qquad {}_n(e\, t \quad) \qquad n$$

$$\frac{\Gamma(n + a + 1)}{\quad} \quad \frac{(-t)^k}{\quad}$$

Using Leibniz's formula $\qquad 0 \le t <$

$$\infty\;\; \Gamma(k + a + 1\; k\; n!(- k)!$$

$$_n$$

$$L\, t^a{}_n(\,) = \sum_{k=0}$$

$n!\; dt$

For $a = 0$, $L\, t^a{}_n(\,)$ become $L\, t_n(\,)$

23.4.2 Polynomials

$$L\, t^a{}_0(\,) = 1,\; L\, t_1{}^a(\,) = 1 + a - t, L\, t^a{}_2(\,)$$

=

23.5 Recurrence Relations

1. $L^a{}_n{}'(\,)t - L^a{}_{n-}{}'{}_1(\,)t + L^a{}_{n-1}(\,)t = 0 \quad n \;\; = 1 \;\; 2, \; ,\cdots \; n \;\; = 1$
2. $t\, L^a{}_n{}'(\,)t = nL\, t^a{}_n(\,) - (n + a\, L) \;\; {}^a{}_{n-1}(\,)t \quad 2, ,\cdots$

23.5.1 Recurence Relations

23.6 Laguerre Series

23.6.1 Orthogonality Relations

$$\int_{}^{\infty}$$

$\frac{1}{2}[(1+a)(2+a) - 2\,2(+a\,t) + t^2],\ldots$

$e\,t\,L^{t\,aa}{}_m(\,)t\,L\,t\,dt^a{}_n(\,) \quad = 0 \quad a > -1$

$n \neq m$

$$(n+a+1) \quad \int^{\infty} e^{-t} t^a [L_n^a(t)]^2\, dt = \Gamma \quad \frac{}{n\ !} \qquad a > -1\ n = 0\ 1\ 2,\ ,\ \cdots$$

23.6.2 Orthonormal Functions

$n = 0\ 1\ 2,\ ,\ \cdots 0 \leq t < \infty$

$\phi_n^a(\,)t = \frac{}{}\Gamma(n + n!a + 1)\ {}^{1\,2/}\ e^{-t/2} t^{a/2} L\ t_{a_n}(\,)$

23.6.3 Series

The Laguerre series is given by

$$f(t) = \sum_{n=0}^{\infty} C_n L_n^a(t) \qquad 0 \leq t < \infty$$

$$\infty \quad = \frac{n!}{\Gamma(n+a+1)} \int \qquad C_n e\ t^{-t\,a} f\ t\ L\ t\ dt(\,)\ {}^a{}_n(\,) \qquad n = 0\ 1$$

$2,\ ,\ \cdots$

Example

The function t^b can be expanded in series

$$t^b = \sum_{n=0}^{\infty} C\ L\ t_n \qquad {}^a{}_n(\,) \qquad b > -\tfrac{1}{2}(a+1)$$

$$C_n = \frac{n!}{\Gamma(n+a+1)}\int_0^\infty t^{b+a} e^{-t} L_n^a(t)\,dt = \frac{n!}{\Gamma(n+a+1)}\int_0^\infty e^{-t} t^{b+a}\,\frac{e^t t^{-a}}{n!}\frac{d^n}{dt^n}\left(t^{n+a}e^{-t}\right)dt$$

$$= \frac{1}{\Gamma(n+a+1)}\int_0^\infty t^b \frac{d^n}{dt^n}\left(t^{n+a}e^{-t}\right)dt = \frac{(-1)^n b(b-1)\cdots(b-n+1)}{\Gamma(n+a+1)}\int_0^\infty e^{-t}t^{b+a}\,dt$$

$$= (-1)^n \frac{\Gamma(b+1)}{\Gamma(n+b+1)\Gamma(b-n+1)}\int_0^\infty e^{-t}t^{(b+a+1)-1}\,dt = (-1)^n \frac{\Gamma(b+1)\Gamma(b+a+1)}{\Gamma(n+b+1)\Gamma(b-n+1)}$$

The steps to find C_n were: a) substitution of (23.4.1), b) integration by parts n times, c) multiplication numerator and denominator by $\Gamma(b-n+1)$. In particular if $b = m =$ *positive integer*

$$t^m = \Gamma(m+a+1)\,m!\sum_{n=0}^{m}\frac{(-1)^n L_n^a(t)}{\Gamma(n+a+1)(m-n)!} \qquad 0 \le t < \infty,\ a > -1 \text{ and } m = 0, 1, 2, \ldots$$

If $a = 0$ we obtain the expansion

$$t^m = \Gamma(m+1)\,m!\sum_{n=0}^{m}\frac{(-1)^m L_n(t)}{n!(m-n)!}$$

Example

The function $f(t) = e^{-bt}$, with $b > -\frac{1}{2}$ and $t > 0$, is expanded as follows

$$C_n = \frac{n!}{\Gamma(n+a+1)}\int_0^\infty e^{-(b+1)t}t^a L_n^a(t)\,dt = \frac{1}{\Gamma(n+a+1)}\int_0^\infty e^{-bt}\frac{d^n}{dt^n}\left(e^{-t}t^{n+a}\right)dt$$

$$\frac{1}{\Gamma(n+a+1)}\int_0^\infty e^{-(b+1)t}\, t^{n+a}\, dt = \frac{b^n}{(b+1)^{n+a+1}} \qquad n = 0, 1, 2, \cdots$$

and thus

$$e^{-bt} = (b+1)^{-a-1}\sum_{n=0}^{\infty}\left(\frac{b}{b+1}\right)^n L_n^a(t) \qquad 0 \le t < \infty$$

For $a = 0$

$$e^{-bt} = (b+1)^{-1}\sum_{n=0}^{\infty}\left(\frac{b}{b+1}\right)^n L_n(t) \qquad 0 \le t < \infty$$

23.7 Tables of Laguerre Polynomials

TABLE 23.1 Properties of the Laguerre Polynomials

1. $L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n!\, t^k}{(k!)^2 (n-k)!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}\binom{n}{k} t^k \qquad n = 0,1,2,\cdots, 0 \le t < \infty$
2. $L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}) \qquad n = 0,1,2\cdots$
3. $L_0(t) = 1$, $L_1(t) = -t+1$, $L_2(t) = \frac{1}{2!}(t^2 - 4t + 2)$, $L_3(t) = \frac{1}{3!}(-t^3 + 9t^2 - 18t + 6)$, $L_4(t) = \frac{1}{4!}(t^4 - 16t^3 + 72t^2 - 96t + 24)$

   $L_n(0) = 1$, $L_n'(0) = -n$, $L_n''(0) = \frac{1}{2}n(n-1)$
4. $(n+1)L_{n+1}(t) + (t-1-2n)L_n(t) + nL_{n-1}(t) = 0 \qquad n = 1,2,3,\cdots$
5. $L_n'(t) - L_{n-1}'(t) + L_{n-1}(t) = 0$
6. $(n+1)L_{n+1}'(t) + (t-1-2n)L_n'(t) + L_n(t) + nL_{n-1}'(t) = 0$
7. $L_{n+1}'(t) = L_n'(t) - L_n(t)$
8. $tL_n'(t) = nL_n(t) - nL_{n-1}(t)$

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1,2,3,⋯ n

= 1,2,3,⋯

n = 1,2,3,⋯

9. $t L t_n''( ) + (1- t L t)_n'( ) + nL t_n( ) = 0,$ Laguerre differential equation

∞

10. $w t x(,) = (1- x)^{-1} \exp$ − ——1 tx − x = $\sum_{n=0} L t x_n( )^n$

generating function

11. $\int^{\infty} {}^{-t} {}_n( ) {}_k( ) = 0$ $\neq$ $e L t L t dt$ k n

12. $\int^{\infty} e^{-t}[L_n(t)]^2 dt = 1$

13. $f t = \sum_{n=0}^{\infty}( )$ $C L t_n$ ${}_n( )$ $0 \le t < \infty$

$C_n = \int^{\infty} {}^{-t} ( ) {}_n( )$ e $f t L t dt$ n = 0,1,2⋯

d_m

14. $L t_{m_n}( ) = -(1)_m$ ——${}_m [L_{n\,m+}( )]t$ m = 0,1,2⋯ dt

n k k

${}^m(-1) (n + m t)!$ ⋯

15. $L t_n( ) = \sum_{k=0}$ ————————$(n - k)!(m + k k)!!$ m = 0,1,2

16.

$(n+1)L_{n+1}^m(t) + (t - 1 - 2n - m)L_n^m(t) + (n+m)L_{n-1}^m(t) = 0$

17. $t L^m{}_n'( )t - nL t^m{}_n( ) + {}^m_{n-1}$ $0(n + m L)$ $( )t =$

TABLE 23.1 Properties of the Laguerre Polynomials (continued)

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18. — t $-m$ —— n $-tnm+$

m 1 d

$L_n( )t = e\, t_n(e\, t) =$ Rodrigues formula $n!$ dt

19. $L_{mn-1}( )t + L_{mn-1}( )t - L_{mn}( )t = 0$

20. $L_{mn}'( )t = -L_{mn-+11}( )t$

m $(n + m)!$

21. $L_n( )0 =$ ———— $n\, m!\,!$

22. $\int_0^\infty e^{-t} t^k L_n(t)\,dt = \begin{cases} 0 & k < n \\ (-1)^n n! & k = n \end{cases}$

23. $L_n(x) * L_k(t) = \int_0^t L_k(x) L_n(t-x)\,dx = \int_0^t L_{n+k}(x)\,dx = L_{n+k}(t) - L_{n+k+1}(t)$

24. $\int_t^\infty e^{-x} L_n^m(x)\,dx = e^{-t}[L_n^m(t) - L_{n-1}^m(t)] \quad m = 0, 1, 2, \cdots$

25. $\int_0^t (t-x)^m L_n(x)\,dx = \frac{m!\,n!}{(m+n+1)!} t^{m+1} L_n^{m+1}(t) \quad m = 0, 1, 2, \cdots$

26. $\int_0^1 x^a (1-x)^{b-1} L_n^a(tx)\,dx = \frac{\Gamma(b)\,\Gamma(n+a+1)}{\Gamma(n+a+b+1)} L_n^{a+b}(t) \quad a > -1, \; b > 0$

27. $\int_0^\infty e^{-t} t^a L_n^a(t) L_k^a(t)\,dt = 0 \quad k \neq n, \; a > -1$

28. $\int_0^\infty e^{-t} t^a [L_n^a(t)]^2\,dt = \frac{\Gamma(n+a+1)}{n!} \quad a > -1$

29. $\int_0^\infty e^{-t} t^{a+1} [L_n^a(t)]^2\,dt = \frac{\Gamma(n+a+1)}{n!}(2n+a+1) \quad a > -1$

30. $L_n^{-1/2}(t) = \frac{(-1)^n}{2^{2n}\,n!} H_{2n}(\sqrt{t})$

31. $L_n^{1/2}(t) = \frac{(-1)^n}{2^{2n+1}\,n!} \frac{H_{2n+1}(\sqrt{t})}{\sqrt{t}}$

32. $f(t) = \sum_{n=0}^\infty C_n L_n^m(t)$

$$C_n = \frac{n!}{\Gamma(n+m+1)} \int_0^\infty e^{-t} t^m f(t) L_n^m(t)\,dt$$

33. $\phi_n^m = \left[\frac{n!}{\Gamma(n+m+1)}\right]^{1/2} e^{-t/2} t^{m/2} L_n^m(t)$, orthonormal sequence, $n = 0, 1, 2 \cdots$

34. $t^p = p! \sum_{n=0}^{p} (-1)^n \binom{p}{n} L_n(t)$

35. $e^{-at} = (a+1)^{-1} \sum_{n=0}^\infty \left(\frac{a}{a+1}\right)^n L_n(t) \quad a > -\frac{1}{2}$

36. $\int_0^\infty \frac{e^{-tx}}{x+1}\,dx = \sum_{n=0}^\infty \frac{L_n(t)}{n+1}$
