ENTRI

Laurent Expansion

- **Laurent Expansion Theorem.** Suppose that f is holomorphic on an open annulus $A(a;r,R)$. Then f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

which converges absolutely in the annulus and uniformly on compact subannuli. Moreover, the coefficients c_n of the Laurent expansion are determined uniquely as

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a}\, d\zeta, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

where γ is any positively oriented Jordan curve in the annulus which wraps around a.

- **Local Laurent expansion.** A Laurent series expansion of f on a "deleted neighborhood" of a, i.e. an annulus of the form $A(a;0,R)$, is called the *local* Laurent expansion of f at a.

- **Laurent estimates.**If $f(z)$ = P$_{n\in Z}$ These are analogs of the Cauchy Estimates for holomorphic functions.$_n$ is a Laurent expansion in the annulus $A(a;R_1,R_2)$, then for every

$c_n(z-a)$

$R_1 < R < R_2$, we have

$$|c_n| \le \frac{M_R}{R^n}, \qquad \text{where } M_R := \max_{|z-a|=R} |f(z)|.$$

- **Useful Laurent series.** In practice, its useful to know a couple of important Laurent (and Taylor) series. Among them

Geometric $(|z|<1)$ $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

z $\quad -z\ 1 - z$ $\quad n=1$ $\quad z$

$$\frac{1}{(1-z)^{k+1}} = \frac{1}{k!}\frac{d^k}{dz^k}\left\{\frac{1}{1-z}\right\}$$

Geometric $(|\ \ | > 1)$ $$\frac{1}{1-} = \frac{1}{(\quad {}^1)} = \sum^{\infty} - \frac{1}{{}_n} = -\frac{1}{} - \frac{1}{{}_2} - \frac{1}{{}_3} - \cdots$$

z z z

z

Derivatives!

Binomial $$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} z^n$$

$(\forall\, z$ $\quad \sum_{n=0} \overline{n}$

Sine $(\forall\, z \in \mathbb{C})$ $$\sin z = \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n-1)!} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$$

Cosine $(\forall\, z \in \mathbb{C})$ $$\cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \cdots$$

Exponential $\in \mathbb{C})$ $$e^z = \sum^{\infty} \frac{z^n}{!} = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \cdots$$

$(|z| < 1)$

Special types of points

- **Zeros.** A *zero* of a nonconstant function f is a point a such that $f(a) = 0$. If f is holomorphic and a is a zero of f, then we can Laurent (or Taylor!) expand f as

$$f(z) = \sum_{n=m}^{\infty} c_n(z-a)^n = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + \cdots .$$

The smallest $m \geq 1$ such that $c_m = 06$ is called the *order* of the zero. Then the following are equivalent:

a is a zero of order $m \Longleftrightarrow f(a) = f^0(a) = \cdots = f^{(m-1)}(0) = 0, f^{(m)}(0) = 06 \Longleftrightarrow$ there exists a *nonzero* analytic g defined near a

s.t. $\quad f(z) = (z-a)^m g(z) \qquad \forall z$ near a

- **Isolated singularities.** A point a is an *isolated singularity* of f if f is *not* differentiable at a, but *is* differentiable on a *deleted neighborhood* of a, i.e. an annulus of the form $A(a;0,R)$. There are three types of isolated singularities classified by the behavior of f near a:

Removable:	$\lim_{z\to a} f(z) = A \in \mathbb{C}$, so f can be made continuous at a
Pole:	$\lim_{z\to a} f(z) = \infty$, so f can be made continuous in $\mathbb{C}_\infty$
Essential:	$\lim_{z\to a} f(z)$ does not exist, so f cannot be made continuous

- **Removable singularities.** The following conditions are also equivalent for an isolated singularity a of a function f:

a is removable	$\Longleftrightarrow$	f can be made continuous at a
	$\Longleftrightarrow$	f is bounded on a neighborhood of a
	$\Longleftrightarrow$	the local Laurent series at a has *no* singular part
	$\Longleftrightarrow$	f can be made analytic at a

- **Poles.** It is easy to see that $f(z)$ has a pole at a if and only if the reciprocal function $\frac{1}{f(z)}$ has a removable zero at a. Hence, every pole has a corresponding *order*. In fact, a is pole of order m $\Longleftrightarrow$ $1/f$ has a removable zero of order m at a

$\Longleftrightarrow$ there exists a *nonzero* analytic g defined near a

s.t. $f(z) = \dfrac{g(z)}{(z-a)^m} \quad \forall z$ near a

$\Longleftrightarrow$ the local Laurent series at a has *finite* singular part

- **Essential singularities.** The following are equivalent for an isolated singularity a:

a is an essential singularity $\Longleftrightarrow$ the local Laurent series at a has *infinite* singular part $\Longleftrightarrow$ for any $A \in \mathbb{C}_\infty$, there exists a sequence z_n

s.t. $z_n \to a, f(z_n) \to A$ as $n \to \infty$