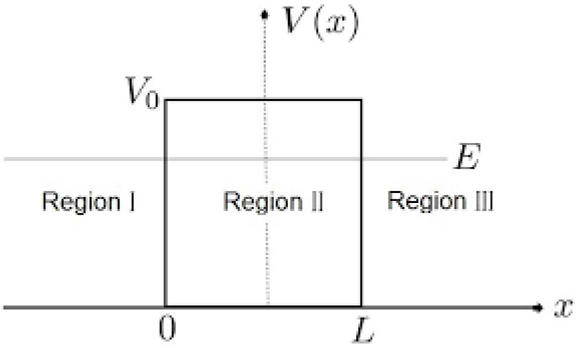
Quantum mechanical tunneling



The potential V(x) is defined by

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 < x < L \\ 0 & x > 0 \end{cases}$$

Consider a stream of particles of mass m approaching the square barrier from the left. Classically the energy of the particle E$< V_0$ it is always reflected where as it is transmitted if E $>V_0$. However, quantum mechanically it can be seen that there is always a finite probability for a particle to penetrate or leak through the barrier and continue its forward motion even if E$< V_0$. this phenomenon is called quantum mechanical tunneling, is possible because of the wave nature of matter.

Case (I) E$< V_0$

The schrödinger equation for region (1) is given by

$$\frac{\partial^2 \Psi_1}{\partial x^2} + \frac{2mE}{\hbar^2} \Psi_1 = 0 \rightarrow (1)$$

Put $k^2 = \frac{2mE}{\hbar^2}$

$$(1) \rightarrow \quad \frac{\partial^2 \Psi_1}{\partial x^2} = -K^2 \Psi_1$$

On solving this $(D^2 + k^2)\Psi_1 = 0$

$$D^2 = -K^2$$

$$D = \pm ik$$

Therefore $\Psi_1 = Ae^{ikx} + Be^{-ikx} \rightarrow (2)$

The first term represents the incident wave and second term represents the reflected wave.

$\Psi_i = Ae^{ikx}$

$\Psi_r = Be^{-ikx}$

In region 2

$\frac{\partial^2 \Psi_2}{\partial x^2} + \frac{2m(V_0 - E)}{\hbar^2} \Psi$ here $\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2}$

On solving this we get

Ψ_2 = B $e^{\propto x}$ + C $e^{-\propto x}$

For x ≥ L

$\frac{\partial^2 \Psi_3}{\partial x^2} = -k^2 \Psi_3 \rightarrow (3)$

The probability density of the incident, reflected and transmitted waves are 1, $|A|^2$ *and* $|D|^2$, respectively. Consequently the transmission coefficient T = $|D|^2$ and the reflection coefficient R= $|A|^2$. The two are connected by the relation R+T = 1. The continuity conditions on the wave functions and their first derivative at x = 0 and x = l give

$1 + A = B + C \rightarrow (4)$

$ik - ikA = \alpha B - \alpha C \rightarrow (5)$

And

B $e^{\propto l}$ + C $e^{-\propto l}$ = D$e^{ikl} \rightarrow (7)$

αB $e^{\propto l}$ - αC $e^{-\propto l}$ = ikD$e^{ikl} \rightarrow (8)$

On solving these equations we get

B = $\frac{D}{2\alpha}(\alpha + ik)e^{ikl - \alpha l} \rightarrow (9)$

C = $\frac{D}{2\alpha}(\alpha - ik)\, e^{ikl + \alpha l} \rightarrow (10)$

On solving (4) and (9) we get,

$$A = \frac{D(\alpha + ik)}{(\alpha - ik)} e^{ikl - \alpha l} - \frac{\alpha + ik}{\alpha - ik}$$

Substituting A , B, C in equ (4) gives

D = $\frac{2ik\alpha e^{-ikl}}{(\alpha^2-k^2)^2 \sin h\,(\alpha l)-2i\alpha k \cosh\,(\alpha l)}$

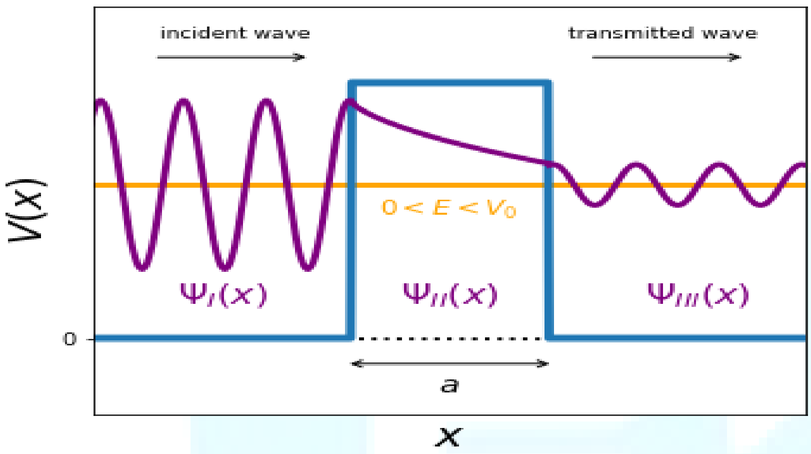
Now T = $|D|^2$ and also assuming that for broad high barrier

$\alpha a >> 1$ and $\sinh(\propto l) = \cosh(\alpha l) \rightarrow \frac{1}{2} e^{l\alpha}$

So T = $\frac{16k^2\alpha^2 e^{-2l\alpha}}{\alpha^2+k^2)^2}$

Substituting the values of $\alpha^2 and\ k^2$ gives

$$T = \frac{16E\,(V_0-E)e^{-2\alpha l}}{{V_0}^2}$$



An illustration of the wave function in three regions