Rodrigues' formula for the Legendre polynomials

1 Introduction

Legendre polynomials $P_n(x)$ are solutions of Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \text{for } n \in \mathbb{N} \cup \{0\} \tag{1}$$

and one explicit, compact expression for the polynomials is by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{2}$$

This means that when $P_n(x)$ is plugged in the position of y for Equation (1), it must satisfy the equality to 0. In this note, we show indeed the expression (2) works, after a bit of tedious arithmetics.

2 Main

I will proceed in two steps. Let $f_n(x) = (x^2-1)^n$ then we first show that the n-th derivative of $f_n(x)$ is a solution of Legendre equation. Then, we find a proper scaling factor of $1/2^n n!$ to recover $P_n(x)$ in line with a common constraint that $P_n(x) = 1$ for all n when $x = 1$. For notational simplicity, we denote $g^{(n)}$ for the n-th derivative of a function $g(x)$, i.e,

$$g^{(n)} = \frac{d^n}{dx^n}g(x).$$

Before proceeding, we need (generalized) Leibniz's rule. Suppose we have n-times differentiable functions $f(x)$ and $g(x)$, then

$$\frac{d^n}{dx^n}f(x)g(x) = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}(x)g^{(k)}(x) = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}f^{(n-k)}(x)g^{(k)}(x) \tag{3}$$

where the choice of f and g can help in reducing the number of terms when there exists a polynomial term. For example, when $g(x) = x^2$, $g^{(k)} = 0$ for all $k \geq 3$.

Part 1. $f_n^{(n)}(x)$ is one solution.

Our goal here is to show that $f_n^{(n)}(x)$ is a solution for Equation (1). As a first step, let's take derivative on $f_n(x)$,

$$\frac{d}{dx}f_n(x) = 2n(x^2-1)^{n-1}x$$

$$= 2nx(x^2-1)^{n-1}$$

and multiply (x^2-1) on both sides

$$(x^2-1)\frac{d}{dx}f_n(x) = 2nx(x^2-1)^n.$$

Now, differentiate both sides $(n+1)$ times, which leads to

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}\left[\frac{d}{dx}f_n(x)\right](x^2-1) &= \sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{d}{dx}f_n(x)\right)^{(n+1-k)}(x^2-1)^{(k)}\\ &= \binom{n+1}{0}f_n^{(n+2)}(x)(x^2-1)+\binom{n+1}{1}2xf_n^{(n+1)}(x)+\binom{n+1}{2}f_n^{(n)}(x)\cdot 2\\ &= (x^2-1)f_n^{(n+2)}(x)+2(n+1)xf_n^{(n+1)}(x)+n(n+1)f_n^{(n)}(x)\end{aligned}$$

for the left-hand side, and

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}f_n(x)2nx &= \binom{n+1}{0}f_n^{(n+1)}(x)2nx+\binom{n+1}{1}f_n^{(n)}(x)2n\\ &= 2nxf_n^{(n+1)}(x)+2n(n+1)f_n^{(n)}(x).\end{aligned}$$

Therefore, we have following arrangement,

$$(x^2-1)f_n^{(n+2)}(x) + 2x(n+1)f_n^{(n+1)}(x) + n(n+1)f_n^{(n)}(x) = 2nxf_n^{(n+1)}(x) + 2n(n+1)f_n^{(n)}(x)$$

$$(x^2-1)f_n^{(n+2)}(x) + 2xf_n^{(n+1)}(x) - n(n+1)f_n^{(n)}(x) = 0$$

$$(1-x^2)f_n^{(n+2)}(x) - 2xf_n^{(n+1)}(x) + n(n+1)f_n^{(n)}(x) = 0$$

where the last line is in the form of Equation (1) so that we have $f_n^{(n)}(x)$ as a solution.

Part 2. find a scaling factor.

Even though $f_n^{(n)}(x)$ as a solution, we have a requirement for the standard Legendre polynomial that $P_n(x) = 1$ for $x = 1$. Let us take a closer look at $f_n^{(n)}(x)$ when evaluated at $x = 1$.

$$\begin{aligned}f_n^{(n)}(x) &= \frac{d^n}{dx^n}(x^2-1)^n\\ &= \frac{d^n}{dx^n}(x+1)^n(x-1)^n\end{aligned}$$

and by Leibniz's rule, we have

$$\begin{aligned}
&= \sum_{k=0}^{n} \binom{n}{k} \left((x+1)^n\right)^{(k)} \left((x-1)^n\right)^{(n-k)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!}(x+1)^{n-k} \frac{n!}{k!}(x-1)^k \qquad (*) \\
&= n! \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!k!}(x+1)^{n-k}(x-1)^k \\
&= n! \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^{n-k}(x-1)^k.
\end{aligned}$$

Since we want to evaluate $f_n^{(n)}(x)$ at x = 1, the last line of equations above tells us that all the terms but k = 0 become zero,

$$f_n^{(n)}(x=1) = n! \binom{n}{0}^2 2^{n-0} = n!2^n$$

which finally leads to define $P_n(x)$ as

$$P_n(x) = \frac{1}{n!2^n} f_n^{(n)}(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}(x^2-1)^n$$

to fulfill the condition of $P_n(x) = 1$ for $x = 1$.