The Calculus of Residues

"Using the Residue Theorem to evaluate integrals and sums"

The residue theorem allows us to evaluate integrals without actually physically integrating i.e. it allows us to evaluate an integral just by knowing the residues contained inside a curve. In this section we shall see how to use the residue theorem to to evaluate certain real integrals which were not possible using real integration techniques from single variable calculus and how to find the values of certain infinite sums. In order to do this, we shall present a number of different types of integrals and sums and then the method of how to calculate them. We shall provide proofs for only certain integrals and sums.

1. Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ for a real variable x with $Q(x) = 06$.

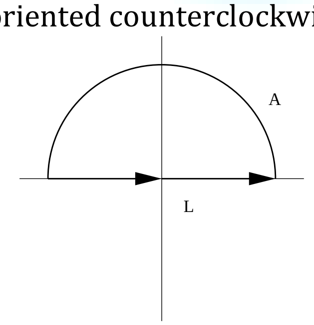
Theorem 1.1. *Suppose that* $P(x)$ *and* $Q(x)$ *are polynomials of a real variable* x *,* $Q(x) = 06$ *for any real* x *and* $\deg(Q(x)) - \deg(P(x)) > 2$ *(to guarantee the convergence of the integral* $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$*). Then*

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_k Res(P/Q; z_k)$$

where the z_i *are the singularities in the upper half plane.*

Proof. We prove this through a number of steps.

(i) Let C denote the curve which consists of A, the semicircle of radius R centered at the origin and L the real line segment from $-R$ to R oriented counterclockwise (as illustrated).



(ii) By the residue theorem,

$$\int_C \frac{P(z)}{Q(z)} dz = 2\pi i \sum_k Res_C(P/Q; z_k)$$

where the z_i are the singularities in the interior of C.

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(iii) Let A denote just the semicircle. Since deg($Q(x)$)−deg ($P(x)$) > 2, then for sufficiently large R, there exists a constant k such that $|P(z)/Q(z)|$ 6 k/R^2 on A. Using the ML formula, it follows that

$$\int_A \frac{P(z)}{Q(z)} dz \; 6 \; \pi R \frac{k}{R^2} = \frac{\pi k}{R} \to 0$$

as $R \to \infty$.

(iv) It follows that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = lim_{R\to\infty}\left(\int_L \frac{P(z)}{Q(z)} dz + \int_A \frac{P(z)}{Q(z)} dz\right)$$

$$= \lim_{R\to\infty} (2\pi i \sum_k Res_C(P/Q; z_k)) = 2\pi i \sum_k Res(P/Q; z_k)$$

where the z_i are the singularities in the upper half plane.

Example 1.2. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2+x}{(x^2+1)(x^2+4)} dx.$$

By the previous result,

$$\int_{-\infty}^{\infty} \frac{x^2+x}{(x^2+1)(x^2+4)} dx = 2\pi i \sum_k Res(P/Q; z_k)$$

where the z_i are the singularities in the upper half plane. The poles in the upper half plane are at $z = i$ and $z = 2i$ and both are simple poles, so we can evaluate them easily. Specifically, if $P(z) = z^2 + z$ and $Q(z) = (z^2 + 1)(z^2 + 4)$, then we have

$$Res(P(z)/Q(z); i) = \frac{P(i)}{Q'(i)} = \frac{i^2+i}{2i(i^2+4)+2i(i^2+1)} = \frac{i-1}{6i}$$

and

$$Res(P(z)/Q(z); 2i) = \frac{P(2i)}{Q'(2i)} = \frac{-4+2i}{4i((2i)^2+4)+4i((2i)^2+1)} = -\frac{-4+2i}{-12i} = \frac{2-i}{6i}.$$

Therefore, we get

$$\int_{-\infty}^{\infty} \frac{x^2+x}{(x^2+1)(x^2+4)} dx = 2\pi i(\frac{i-1}{6i} + \frac{2-i}{6i}) = 2\pi i\frac{1}{6i} = \frac{\pi}{3}$$

2. Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(x) dx$ and $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x) dx$ for a real variable x with $Q(x) = 06$.

Theorem 2.1. *Suppose that* $P(x)$ *and* $Q(x)$ *are polynomials of a real variable* x *,* $Q(x) = 06$ *for any real* x *and* $\deg(Q(x)) - \deg(P(x)) >$ 1 *(to guarantee the convergence of the integral* $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(x) dx$ *and* $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x) dx$*). Then if* $R(z) = P(z)/Q(z)$*, then*

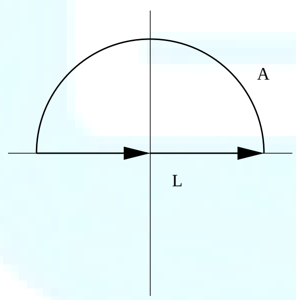
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(x) dx = Im(2\pi i \sum_k Res(R(z)e^{iz}; z_k))$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x) dx = Re(2\pi i \sum_k Res(R(z)e^{iz}; z_k))$$

where the z_i *are the singularities of* $R(z)e^{iz}$ *in the upper half plane.*

Proof. We proceed as with the last case but instead of integrating $R(z)\cos(z)$ or $R(z)\sin(z)$, we consider $R(z)e^{iz}$. Let C denote the curve which consists of A, the semicircle of radius R centered at the origin and L the real line segment from $-R$ to R oriented counterclockwise (as illustrated).



Fix some real $h > 0$. Then we break up the curve A into the segments $A_+ = \{z \in A | Im(z) > h\}$ and $A_- = \{z \in A | Im(z) < h\}$. We also observe the following:

- Since $\deg(Q) - \deg(P) > 1$, we have $|R(z)| > K/|z|$ for some constant K and for sufficiently large z
- $|e^{iz}| = e^{Re(iz)} = e^{-y}$ for $z = x + iy$.

Using these observations, we have the following:

(i)

$$\left|\int_{A_+} R(z)e^{iz}dz\right| \leqslant K\frac{e^{-h}}{R}\pi R = K\pi e^{-h}$$

and $\left|\int_{A_-} R(z)e^{iz}dz\right| \leqslant \frac{K}{R}4h = 4\frac{Kh}{R}$

since clearly each segment making up A_- have lengths less than $2h$.

(ii) Choosing $h = \sqrt{M}$, we have

$$\left|\int_{A_+} R(z)e^{iz}dz\right| \leqslant K\pi e^{-\sqrt{R}}$$

and $\left|\int_{A_-} R(z)e^{iz}dz\right| \leqslant 4K\frac{\sqrt{R}}{R} = \frac{4K}{\sqrt{R}}$

so it follows that $\left|\int_{A} R(z)e^{iz}dz\right| \leqslant K\pi e^{-\sqrt{R}} + \frac{4K}{\sqrt{R}} \to 0$

as $R \to \infty$. Thus

$$\lim_{R\to\infty}\int_C R(z)e^{iz}dz = \int_{-\infty}^{\infty} R(x)e^{ix}dx.$$

(iii) It follows that

$$\sum_k 2\pi i Res_C(R(z)e^{iz};z_k) = \lim_{R\to\infty}\int_C R(z)e^{iz}dz = \int_L R(x)e^{ix}dx$$

$$= \int_L R(x)(\cos(x)+i\sin(x))dx = \int_L R(x)\cos(x)dx + i\int_L R(x)\sin(x)dx$$

by the residue theorem where the z_i are the singularities of $R(z)e^{iz}$ in the upper half plane. Hence

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin(x)dx = Im(2\pi i\sum_k Res(R(z)e^{iz};z_k))$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x)dx = Re(2\pi i \sum_k Res(R(z)e^{iz}; z_k))$$

We illustrate again with an example.

Example 2.2. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9}dx$$

and

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+9}dx.$$

By the previous result,

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9}dx = Re(2\pi i \sum_k Res(\frac{e^{iz}}{z^2+9}; z_k))$$

where the z_i are the singularities in the upper half plane. There is a single simple pole in the upper half plane at $z = 3i$ so we can evaluate the residue easily. Specifically, if $P(z) = e^{iz}$ and $Q(z) = z^2 + 4$, so

$$Res(P(z)/Q(z); 3i) = \frac{P(i)}{Q'(i)} = \frac{e^{-3}}{6i}.$$

Therefore, we get

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9}dx = \frac{\pi e^{-3}}{3}$$

and

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+9}dx = 0.$$

3. Integrals of the form $\int_0^{\infty} \frac{P(x)}{Q(x)}dx$ for a real variable x with $Q(x) = 06$.

First note that if $P(x)/Q(x)$ is an even function, we can use the previous result and the fact that

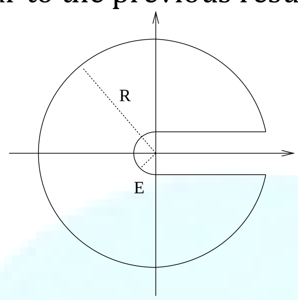
$$\int_0^\infty \frac{P(x)}{Q(x)}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{P(x)}{Q(x)}dx.$$

Otherwise we integrate the function $R(z)\log(z)$ where $R(z) = P(z)/Q(z)$ over the following contour C and note that as $R \to \infty$ and $\varepsilon \to 0$, we

have

$$\lim_{R\to\infty}\int_C R(z)\log(z)dz = -\frac{1}{2\pi i}\int_0^\infty \frac{P(x)}{Q(x)}dx$$

using calculations similar to the previous results.



Therefore, we get the following:

Theorem 3.1. *Suppose that* $P(x)$ *and* $Q(x)$ *are polynomials of a real variable* x *,* $Q(x) = 06$ *for any real* x *and* $\deg(Q(x)) - \deg(P(x)) > 2$ *(to guarantee the convergence of the integral* $\int_0^\infty \frac{P(x)}{Q(x)}dx$*). Then*

$$\int_0^\infty \frac{P(x)}{Q(x)}dx = -\sum_k Res(\frac{P(z)}{Q(z)}\log(z); z_k)$$

where the z_i *are the singularities in the the whole complex plane.*

Rather than providing a complete proof, we illustrate with an example.
Example 3.2. Evaluate

$$\int_0^\infty \frac{x}{x^4+1}dx.$$

Here we have

$$\int_0^\infty \frac{x}{x^4+1}dx = -\sum_k Res(\frac{z}{z^4+1}\log(z); z_k)$$

where the z_i are the singularities in the the whole complex plane. The poles are all simple and are at the solutions of $z^4 = -1$ i.e. $z = e^{i\pi/4}$, $e^{3i\pi/4}$, $e^{5i\pi/4}$ and $e^{7i\pi/4}$. Evaluating the residues, we have

$ki\pi/4) = QP'((ee^{ki\pi/ki\pi/44}))$ ———— $\log(e^{ki\pi/4})$ ———— $=$

$4ee^{ki\pi/3ki\pi/44}\log(e^{ki\pi/4})$

$Res(P(z)/Q(z)\log(z);e$

1 $k\pi i$ $k\pi i$

$= 4e^{ki\pi/2}$ 4 $= 16e^{ki\pi/2}$

for k = 1,3,5,7. Thus we have

$$\int_0^\infty \frac{x}{x^4+1}dx = -\left(\frac{\pi i}{16e^{i\pi/2}} + \frac{3\pi i}{16e^{3i\pi/2}} + \frac{5\pi i}{16e^{5i\pi/2}} + \frac{7\pi i}{16e^{7i\pi/2}}\right)$$

$$= -\left(\frac{\pi i}{16i} - \frac{3\pi i}{16i} + \frac{5\pi i}{16i} - \frac{7\pi i}{16i}\right) = \frac{4\pi}{16} = \frac{\pi}{4}$$

4. Integrals of the form $\int_0^\infty \frac{x^\alpha}{P(x)}dx$ for a real variable x with $P(x)$ = 06 and with $0 < \alpha < 1$.

In order to integrate functions of this type, we use the same contour as the last example. Though it is a little more complicated to show, we get the following result:

Theorem 4.1. *Suppose α is a number with $0 < \alpha < 1$, $P(x)$ is a polynomial of a real variable x , $P(x)$ = 06 for any real x and $\deg(P(x)) > 1$ (to guarantee the convergence of the integral $\int_0^\infty \frac{x^\alpha}{P(x)}dx$). Then*

$$\int_0^\infty \frac{x^\alpha}{P(x)}dx = \frac{2\pi i}{1-e^{2\pi i(\alpha-1)}} \sum_k Res(\frac{z^{\alpha-1}}{P(z)}; z_k)$$

where the z_i are the zeros of $P(z)$.

We illustrate with an example.

Example 4.2. Evaluate

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx.$$

Using the theorem, we have

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{x^2+1}dx = \frac{2\pi i}{1-e^{-\pi i}} \sum_k Res(\frac{z^{-\frac{1}{2}}}{(z^2+1)}; z_k)$$

where the z_i are the zeros of $z^2 + 1$. The zeros of $z^4 + 1$ are $z = \pm i$. They are both simple poles, so we get

$$Res(\frac{z^{-\frac{1}{2}}}{z^2+1}; i) = \frac{i^{-\frac{1}{2}}}{P'(i)} = \frac{e^{-i\pi/4}}{2e^{\pi i/2}} = \frac{1}{2e^{\frac{3i\pi}{4}}} = \frac{e^{\frac{5i\pi}{4}}}{2} = \frac{\sqrt{2}(-1-i)}{4}$$

and

$$Res(\frac{z^{-\frac{1}{2}}}{z^2+1}; -i) = \frac{(-i)^{-\frac{1}{2}}}{P'(-i)} = \frac{e^{-3i\pi/4}}{2e^{3\pi i/2}} = \frac{1}{2e^{\frac{i\pi}{4}}} = \frac{e^{\frac{7i\pi}{4}}}{2} = \frac{\sqrt{2}(1-i)}{4}.$$

Thus we have

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{x^2+1}dx = \frac{2\pi i}{1-e^{-\pi i}} \sum_k Res(\frac{z^{-\frac{1}{2}}}{(z^2+1)}; z_k) = -\pi i\frac{2\sqrt{2}i}{4} = \frac{\sqrt{2}\pi}{2}$$

5. Integrals of the form $\int_0^{2\pi} R(\cos(\vartheta), \sin(\vartheta))d\vartheta$ for a Rational function R and a real variable ϑ.

In order to integrate functions of this type, first observe that

$$\int_0^{2\pi} R(\cos(\vartheta), \sin(\vartheta))d\vartheta = \int_{|z|=1} R(\cos(\vartheta), \sin(\vartheta))\frac{dz}{iz}$$

$$= \int_{z|=1} R\left(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z+\frac{1}{z})\right)\frac{dz}{iz}$$

on $|z| = 1$ since if we put $z = e^{i\vartheta}$, then $\cos(\vartheta) = \frac{1}{2}(z+\frac{1}{z})$ and $\sin(\vartheta) = \frac{1}{2i}(z+\frac{1}{z})$ and $d\vartheta = dz/iz$. This implies the following method to calculate such integrals:

Theorem 5.1.

$$\int_0^{2\pi} R(\cos(\vartheta), \sin(\vartheta))d\vartheta = 2\pi \sum_k Res\left(\frac{R(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z+\frac{1}{z}))}{z}; z_k\right)$$

where the z_k *are the poles in the circle* $|z|$ 6 1.

We illustrate with an example. **Example 5.2.** Evaluate

$$\int_0^{2\pi} \frac{1}{a+\cos(\vartheta)}d\vartheta$$

where $a > 1$ (so that the integrand is always finite).
We evaluate using the theorem. We have

$$\int_0^{2\pi} R(\cos(\vartheta), \sin(\vartheta))d\vartheta = 2\pi \sum_k Res\left(\frac{\frac{1}{a+\frac{1}{2}(z+\frac{1}{z})}}{z}; z_k\right).$$

However, observe that

$$\frac{\frac{1}{a+\frac{1}{2}(z+\frac{1}{z})}}{z} = \frac{2}{z^2+2az+1}$$

which has poles at $z = -a \pm \sqrt{a^2 - 1}$, which are both on the real axis.

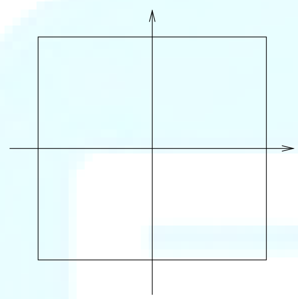
$\sqrt{a^2-1} < a$, so $-$ $-$ Next note that $\sqrt{a^2-1}$ is outside the unit circle. Thus we have $z = -a < +\sqrt{a}a+^2$ $-\sqrt{1}$ a^2 is inside the unit circle since $-1 < 0$ and similarly, $z = a - -a1$ $-<$

$$\int_0^{2\pi} R(\cos(\vartheta), \sin(\vartheta))d\vartheta = 2\pi Res\left(\frac{2}{z^2+2az+1}; -a+\sqrt{a^2-1}\right)$$
$$= 2\pi \frac{2}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

6. Sums of Series

The last application of the residue theorem is to the number theoretical problem of summing series. In order to evaluate sums of series, we shall construct functions whose residues agree with the terms of the series and then integrate over a appropriately chosen contour and apply the residue theorem. Most of the integrals we shall perform will be over the square C_N centered at the origin with vertices $(\pm(N + 1/2), \pm(N + 1/2))$

(see illustration below).



In order to do this, we need the following two results.

Lemma 6.1. *For any $N > 0$, for any z on C_N we have*

$$|\cot(\pi z)| \; 6 \; \coth(\pi/2)$$

and

$$|\mathrm{cosec}(\pi z)| \; 6 \; 1.$$

Proof. Putting $z = x + iy$ and using the trigonometric sum formulas and basic identities, we have

$$|\cos(\pi z)|^2 = \cos^2(\pi x) + \sinh^2(\pi y)$$

and

$$|\sin(\pi z)|^2 = \cosh^2(\pi y) - \cos^2(\pi x).$$

It follows that

$$|\cot(\pi z)|^2 = \frac{\cos^2(\pi x) + \sinh^2(\pi y)}{\cosh^2(\pi y) - \cos^2(\pi x)}.$$

On the vertical sides of C_N, we have

$$x = \pm(N + \frac{1}{2})$$

giving

$$\cos\left((N+\frac{1}{2})\pi\right)=0$$

so

|cot(*πz*)| = |tanh(*πy*)| 6 1*.*

On the horizontal sides,

0 6 cos^2 (*πx*) 6 1

so

$$|\cot(\pi z)|^2 \leqslant \frac{\sinh^2(\pi y)+1}{\cosh^2(\pi y)-1} = \frac{\cosh^2(y)}{\sinh^2(\pi y)} = \coth^2(\pi y).$$

Then we have

|cot(*πz*)| 6 |coth(*πz*)| = coth($\pi(N+\frac{1}{2})) \leqslant$ coth($\frac{\pi}{2}$)*.*

Hence on C_N, we have

|coth(*πz*)| 6 max(1,coth($\frac{\pi}{2}$)) = coth($\frac{\pi}{2}$)*.*

We use a similar argument with cosec(*πz*) to show that on the vertical sides we have

$$|\mathrm{cosec}(\pi z)| \leqslant \frac{1}{|\cosh(\pi y)|} \leqslant 1$$

and on the horizonal sides giving

$$|\mathrm{cosec}(\pi z)| \leqslant \frac{1}{|\sinh(\frac{\pi}{2})|} \leqslant 1$$

the result.

Lemma 6.2. (*i*)

*π*cosec(*π*z)

has a simple pole at z = *n with residue* $(-1)^n$ *for all integers*

(ii) n *.*

π coth(*πz*)

has a simple pole at z = *n with residue* 1 *for all integers n.*

Proof. Clearly

$$\pi\mathrm{cosec}(\pi z) = \frac{\pi}{\sin(\pi z)}$$

and

$$\pi \coth(\pi z) = \frac{\pi \cosh(\pi z)}{\sinh(\pi z)}$$

have isolated singularities at $z = n$ for each integer n and through direct calculation, we can show that these are the only singularities of these functions (by simply finding when $\sin(\pi z) = 0$). Therefore, we just need to determine the residues of these singularities. For any integer n, put $w = z - n$. Then we have $\cos(\pi z) = \cos(\pi w + \pi n) = \cos(\pi w)\cos(\pi n) - \sin(\pi w)\sin(\pi n)$

$$= \cos(\pi w)\cos(\pi n)$$

and $\sin(\pi z) = \sin(\pi w + \pi n) = \sin(\pi w)\cos(\pi n) + \cos(\pi w)\sin(\pi n)$

$$= \sin(\pi w)\cos(\pi n).$$

Next observe that

$$\pi \cot(\pi z) = \frac{\pi \cos(\pi w)}{\sin(\pi w)} = \frac{\pi \cos(\pi w)\cos(\pi n)}{\sin(\pi w)\cos(\pi n)} = \frac{\pi \cos(\pi w)}{\sin(\pi w)}$$

$$= \frac{\pi(1 - \frac{(\pi w)^2}{2!} + \frac{(\pi w)^4}{4!} + \dots)}{(\pi w - \frac{(\pi w)^3}{3!} + \frac{(\pi w)^5}{5!} + \dots)} = \frac{1}{w}\left(\frac{1 - \frac{(\pi w)^2}{2!} + \frac{(\pi w)^4}{4!} + \dots}{1 - \frac{(\pi w)^2}{3!} + \frac{(\pi w)^4}{5!} + \dots}\right).$$

Note that

$$\left(\frac{1 - \frac{(\pi w)^2}{2!} + \frac{(\pi w)^4}{4!} + \dots}{1 \quad \frac{(\pi w)^2}{3!} + \frac{(\pi w)^4}{5!} + \dots}\right)$$

is an analytic function in w, so has a Taylor expansion around $w = 0$ with first term 1 (since plugging $w = 0$ yields 1). Thus we have

$$\frac{1}{w}\left(\frac{1 - \frac{(\pi w)^2}{2!} + \frac{(\pi w)^4}{4!} + \dots}{1 - \frac{(\pi w)^2}{3!} + \frac{(\pi w)^4}{5!} + \dots}\right) = \frac{1}{w}\left(1 + \text{positive powers of } w\right.$$

In particular, the Laurent expanion of z around $z = n$ will have a simple pole with residue 1. A similar argument holds for $\pi \coth(\pi z)$.

Lemma 6.3. *The first three terms of the Laurent expansion of* cosec(z) = 1/sin(z) *around* $z = 0$ *are*

$$\frac{1}{\sin(z)} = \frac{1}{z}\left(1 + \frac{z^2}{6} + \frac{7z^4}{360} + \dots\right).$$

Proof. Since the Taylor series of $\sin(z)$ around $z = 0$ is

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)$$

we have

$$\frac{1}{\sin(z)} = \frac{1}{z}\left(\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots}\right) = \frac{1}{z}\left(\frac{1}{1-\alpha}\right)$$

where

$$\alpha = \frac{z^2}{3!} - \frac{z^4}{5!} + \ldots.$$

Next observe that for sufficiently small z (say $|z| < \delta$), $|\alpha| < 1$ (since it is a continuous function of z which is 0 at 0). Thus for $|z| < \delta$ we have

$$\frac{1}{\sin(z)} = \frac{1}{z}\left(\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots}\right) = \frac{1}{z}\left(\frac{1}{1-\alpha}\right)$$

$$= \frac{1}{z}\left[1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right)^2 + \ldots\right]$$

$$= \frac{1}{z}\left[1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{3!^2} + \text{higher order terms}\right] = \frac{1}{z}\left(1 + \frac{z^2}{6} + \frac{7z^4}{360} + \ldots\right).$$

With these results in consideration, we can use them to evaluate certain special types of sums. Specifically, for a rational function $f(z)$, we do the following:

(i) To evaluate the sum

$$\sum_{n=1}^{\infty} f(n)$$

we consider the integral

$$\int_{C_N} f(z)\pi\cot(\pi z)\,dz$$

and let $N \to \infty$

(ii) To evaluate the sum

$$\sum_{n=1}^{\infty} (-1)^n f(n)$$

we consider the integral

$$\int_{C_N} f(z)\pi\operatorname{cosec}(\pi z)\,dz$$

and let $N \to \infty$

We illustrate with some examples.

Example 6.4. (*i*) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We consider the integral

$$\lim_{N\to\infty} \int_{C_N} \frac{\pi \cot(\pi z)}{z^2} dz.$$

By the residue theorem, we have

$$\lim_{N\to\infty} \int_{C_N} \frac{\pi \cot(\pi z)}{z^2} dz = 2\pi i \sum (\text{ residues in } C_N) = 2\pi i \sum_{n=-N}^{N} (\text{ residues at } z = n)$$

so we need to evaluate the residues of

$$\frac{\pi \cot(\pi z)}{z^2}$$

in C_N. First observe that $1/z^2$ is analytic away from $z = 0$, so for z =6 n,

$$\frac{\pi \cot(\pi z)}{z2}$$

will have a simple pole (since $\cot(\pi z)$ has a simple pole). To calculate the value of the residue, since $\cot(\pi z)$ has a simple pole at $z = n$ with residue 1 because the value of $1/z^2$ at $z = n$ is 1, the power series representation will be

$$\frac{\pi \cot(\pi z)}{z^2} = \left(\frac{1}{n^2} + \text{higher powers of } (z-n)\right)\left(\frac{1}{z-n} + \text{higher powers of } (z-n)\right)$$

which has residue 1. Thus the sum of the residues for z = 06 will be

$$\sum_{n=-N, n\neq 0}^{N} \frac{1}{n^2} = 2\sum_{n=1}^{N} \frac{1}{n^2}.$$

At $z = 0$, we have

$$\pi \frac{\cot(\pi z)}{z^2} = \frac{\pi}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)} = \pi \cos(\pi z)\operatorname{cosec}(\pi z)$$

$$= \frac{1}{z^3}\left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \dots\right)\left(1 + \frac{\pi^2 z^2}{6} + \frac{7\pi^4 z^4}{360} - \dots\right).$$

Expanding, we see that the residue is

π2 π2 π2 − 2 6 − 3

\+ = .

Thus using the residue theorem, we have

$$\lim_{N\to\infty} \int_{C_N} \frac{\pi \cot(\pi z)}{z^2} dz = 2\pi i \sum (\text{ residues in } C_N) = 2\pi i\left(-\frac{\pi^2}{3} + 2\sum_{n=1}^{N} \frac{1}{n^2}\right).$$

Finally, we consider the integral

$$\int_{C_N} \frac{\pi \cot(\pi z)}{z^2} dz.$$

Observe that at any point on the boundary, we have

$$\left| \frac{\pi \cot(\pi z)}{z^2} \right| \leqslant \frac{\pi \coth\left(\frac{\pi}{2}\right)}{(N+\frac{1}{2})^2}$$

so using the *ML*-formula, we have

$$\int_{C_N} \frac{\pi \cot(\pi z)}{z^2} dz \leqslant \frac{\pi \coth\left(\frac{\pi}{2}\right)}{(N+\frac{1}{2})^2} 8\left(N+\frac{1}{2}\right) = \frac{8\pi \coth\left(\frac{\pi}{2}\right)}{N+\frac{1}{2}} \to 0$$

as $N \to \infty$. Thus

$$2\pi i\left(-\frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{1}{n^2}\right) = 0$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Example 6.5. Evaluate

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}.$$

For this case, we observe that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4} = -\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n^4},$$

so we shall consider the integral

$$\lim_{N\to\infty} \int_{C_N} \frac{\pi \operatorname{cosec}(\pi z)}{z^4} dz$$

and negate the answer. As with the previous case, we see that

$$\int_{C_N} \frac{\pi \operatorname{cosec}(\pi z)}{z^4} dz \leqslant \frac{\pi}{(N+\frac{1}{2})^4} 8\left(N+\frac{1}{2}\right) = \frac{8\pi}{(N+\frac{1}{2})^3} \to 0$$

as $N \to \infty$, so it follows that

$$\sum_{n=-\infty}^{\infty} (\text{ residues at } z = n) = 0.$$

For each integer $n = 06$, the residue is $(-1)^n/n^4$, so it follows that the sum of all residues except at $z = 0$ will be

N $\quad \overline{-\quad n} \quad$ $^N(-1)^n$

(1)

$4 \quad = 2X \quad n4 \quad . \overline{n}$

X

$n=-N,n6=0$ $n=1$ At $z = 0$ we

have

$$\frac{\pi \operatorname{cosec}(\pi z)}{z^4} = \pi \frac{1}{z^4}\frac{1}{\pi z}\left(1+\frac{\pi^2 z^2}{6}+\frac{7\pi^3 z^4}{360}-\dots\right) = \frac{1}{z^5}\left(1+\frac{\pi^2 z^2}{6}+\frac{7\pi^4 z^4}{360}-\dots\right)$$

so the residue is

$$\frac{7\pi^4}{360}.$$

It follows that

$7\pi^4$ X^N $(-1)^n$

$+ 2 = 0$

$360\, n^4$ $n=1$ so

$-7\pi^4$ X^N $(-1)^n$

=

720 n^4 $n=1$

giving

$$\frac{7\pi^4}{720} = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^4}.$$