# The Hilbert Space

In a vector space $e_1, e_2, e_3 \ldots.$ form an orthonornal basis. Alternately, a space can be defined in which a set of functions $\emptyset_1(x), \emptyset_2(x), \emptyset_3(x) \ldots.$ form an orthonormal unit vectors of the coordinate system. The corresponding infinite - dimensional linear space is called function space. In quantum mechanics, very often we deal with complex functions and the corresponding function space is called the Hilbert space.

## Orthogonal functions

Consider the important definitions regarding orthogonal functions.

(i) The inner product or scalar product of two functions F(x) and G(x) defined in the interval $a \leq x \leq b$, defined by (F, G) or (F|G) , is

$$(F, G) = \int_a^b F^*(x)\, G(x)\, dx$$

The notation (F, G ) for the scalar product of functions F(x) and G(x) is sometimes referred as bracket notation.

(ii) Two functions F(x) and G(x) are orthogonal if their inner product is zero.

$$(F, G) = \int_a^b F^*(x)\, G(x)\, dx = 0$$

(iii) The norm of a function is defined by square root of inner product of the function with itself.

$$N = (F, F)^{1/2} = \left[\int_a^b |F(x)|^2 dx\right]^{1/2}$$

(iv) A function is normalized if its norm is unity.

$$(F, F)^{1/2} = \left[\int_a^b |F(x)|^2 dx\right]^{1/2}$$

Or $$(F, F) = \int_a^b F^*(x) F(x) dx = 1$$

Where the integral on the right hand side is called normalization integral.

(v) Functions that are orthogonal and normalized are called orthonormal functions.

$$(F_i, F_j) = \delta_{ij} \qquad \text{i, j = 1,2, . . . .}$$

(vi) A set of functions $F_1(x), F_2(x), F_3(x)$ ,…. is linearly independent if a relation of the type.

$$\sum_i c_i F_i(x) = 0$$

Exists, where $c_i's$ are not all zero. Otherwise they are linearly independent.

(vii) A set of linearly independent functions $F_1(x), F_2(x), F_3(x)$ ,…. is complete, if there is no other function which falls in the set of linearly independent functions.

The expansion theorem states that any function Ø(x) defined in the same interval can be expanded in terms of the set of linearly independent functions as

$$\emptyset(x) = \sum_i c_i F_i(x)$$

The complete set not be orthonormal. However it is convenient to use orthonormal sets. In such case the coefficients are given by

$$c_i = (F_i , \emptyset )$$

The expansion of a function in terms of a complete orthonormal set of functions is of fundamental importance in quantum mechanics.