Vector space

Vector in a Three - dimensional space

A Vector in a Three - dimensional space is a physical quantity having both magnitude and direction. A vector of magnitude I is called unit vector. To represent an arbitrary vector, we introduce unit vectors $\boldsymbol{e_1}, \boldsymbol{e_2}$ *and* $\boldsymbol{e_3}$ along the positive direction of a right - handed system OXYZ of three mutually perpendicular axes. Then any vector **a** can be expressed as

$$\mathrm{a} = \boldsymbol{a_1 e_1} + \boldsymbol{a_2 e_2} + \boldsymbol{a_3 e_3}$$

Where $\boldsymbol{a_1, a_2, a_3}$ are scalars. The unit vectors $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ are said to form a basis for the set of all vectors in three dimensions. Since these are unit vectors along mutually perpendicular directions, we say that $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ form an orthonormal basis. The scalars $\boldsymbol{a_1, a_2, a_3}$ are the components of a in the basis ($\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$). we may represent a as a vector:

$$\mathrm{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

The components of e_1 are 1,0,0 ; those of e_2 are 0, 1, 0 and those of e_3 are 0, 0, 1 . Hence on expressing them as column vectors, we have

$$\boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The totality of vectors in three dimensional space is called a three dimensional space. Given two vectors.

$$\mathrm{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad \mathrm{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

In this space, the scalar product or inner product of **a** and **b**, denoted by **a . b** or **(a, b)** is defined as

$$(\mathrm{a, b}) = \sum_{i=1}^{3} \boldsymbol{a_i b_i}$$

Vectors in an n - dimensional space

We now generalize these concepts to an n-dimensional real space, with the vectors $\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots\ldots \boldsymbol{e_n}$ forming an orthonormal basis A vector a can be expressed in this orthonormal basis as

$$a = \sum_{i=1}^{n} \boldsymbol{a_i e_i}$$

The totality of all n-dimensional vectors is called an n-dimensional space, or simply an n-vector space. In this space, the inner product is defined as

$$(a, b) = \sum_{i=1}^{3} \boldsymbol{a_i b_i}$$

If the vectors are complex, that is, if the component of the vectors are complex numbers, the inner product is defined as

$$(a, b) = \sum_{i=1}^{n} \boldsymbol{a_i}^* \ \boldsymbol{b_i}$$

Where, for any complex number z, z^* denotes the conjugate. For any vector a

$$(a, a) = \sum_{i=1}^{n} \boldsymbol{a^*}_i \ \boldsymbol{a_i} = \sum_{i=1}^{n} |\boldsymbol{a_i}|^2$$

We can define the norm (N) of a vector a by

$$N = (\boldsymbol{a}, \boldsymbol{a}\ )^{1/2}$$

A vector whose norm is unity is said to be normalized. Thus for a normalized vector a , we have

$$(a, a) = \sum_{i=1}^{n} \boldsymbol{a^*}_i \ \boldsymbol{a_i} = \sum_{i=1}^{n} |\boldsymbol{a_i}|^2 = 1$$

Two vectors a and b are said to be orthogonal if

$$(a\ , b\ ) = 0$$

Using the concept of inner product, we say that the vectors $a_1, a_2, \ldots$
form an orthonormal set if and only if

$$(\boldsymbol{a_i}, \boldsymbol{a_j}) = \boldsymbol{\delta_{ij}} \qquad \textbf{I,j = 1,2, ...}$$

Where δ_{ij} is called Kronecker delta defined by

$$\boldsymbol{\delta_{ij}} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

In an n-dimensional space, a set of n vectors $a_1, a_2, \ldots$ are said to be linearly dependent if there exist scalars $c_1, c_2, \ldots c_n$ not all zero such that

$$\sum_{i=1}^{n} \boldsymbol{c_i a_i} = \mathbf{0}$$

The vectors are linearly independent if $c_1 = c_2 = \ldots = c_n = 0$

A set of n linearly independent vectors $a_1, a_2, \ldots a_n$ spans an n-dimensional space and any vector $\emptyset$ which lies in the space can be expressed in the form.

$$\emptyset = \sum_{i=1}^{n} \boldsymbol{c_i a_i}$$

This set of vectors is complete if there is no other vector which falls in this set of linearly independent vectors.

The generalization of these concepts to an infinite - dimensional space called vector space is straight forward.