7. Cauchy's integral theorem and Cauchy's integral formula

7.1. Independence of the path of integration

Theorem 6.3. can be rewritten in the following form:

Theorem 7.1 : *Let* D *be a domain in* C *and suppose that* $f \in C(D)$*.*
Suppose further that $F(z)$ *is a continuous antiderivative of* $f(z)$ *through* D
D*. Let* z_0 *and* z_T *be distinct points in* D*. Then the integral*

$$\int_{z_0}^{z_T} f(z)dz$$

does not depend on the path of integration, e.g., for every smooth contour $\gamma \subset D$ *which start at* z_0 *and terminates at* z_T*, we have*

$$\int_{z_0}^{z_T} f(z)dz = \int_\gamma f(z)dz = F(z_T) - F(z_0)$$

Theorem 7.1. is called Theorem on the depend of the path of integration. From this theorem we get the following obvious consequence:

Corollary 7.2. : *Under the conditions on* f *of Theorem 7.1., let* γ *be a smooth closed contour which lies entirely in* D*.*[1] *Then*

$$\int_\gamma f(z)dz = 0$$

Our coming considerations are based on the following theorem:

Theorem 7.3. *Let* D *be a domain in* C *and* $f \in C(D)$*. Then the following statements are equivalent:*

(1) f *has a continuous antiderivative in* D*;*

(2)

$$\int_\gamma f(z)dz = 0$$

[1] We call such contours *loops*.

for every loop γ lying in $\subset$ D*.*

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(3) The integral $\int_{z_1}^{z_2} f(z)dz$

is independent of the path of integration; e.g., if γ_1 and γ_2 share the same initial and terminal points, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof: Since the implications 1) → 2) and 2) → 3) already established (Theorem 7.1 and Theorem 7.2), we will concentrate on the proof of 3) → 1).

Select an arbitrary point $z_0 \in$ D and let $z \in$ D. Set

$$F(z) := \int_{z_0}^{z} f(z)dz.$$

We claim that $F(z)$ is an antiderivative of f in D. Before, we notice that the integral is well defined - because of the connectedness of the domain D there ix a contour which combines z_0 and z.

We shall show that

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} \to f(z), \Delta z \to 0.$$

Indeed,

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = \frac{\int_z^{z+\Delta z}(w)dw}{\Delta z},$$

where we integrate along a segment lying completely in the domain.

Regarding Theorem 6.4, we may write

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| =$$

$$= \left|\frac{\int_z^{z+\Delta z}(f(w)-f(z)dw}{\Delta z}\right| \le \|f(w)-f(z)\|_{[z,z+\Delta z]} \to 0 \text{ as } \Delta z \to 0.$$

Thus $F'(z) = f(z)$. This concludes the proof. **Q.E.D.**

7.2. Continuous deformations of loops.

Definition: The loop γ_1 is said to be continuously deformable to the loop γ_2 in the domain D, if there exists a function $z(s,t)$, $(s,t) \in ([0,1] \times [0,1])$ that satisfies the conditions:

1. $z(s,t) \in C^2([0,1] \times [0,1])$;
2. For each fixed $s \in [0,1]$ the function $z(s,t)$ parametrizes a loop in D;
3. The function $z(0,t)$ parametrizes γ_1;
4. The function $z(1,t)$ parametrizes γ_2. **Example:** THE function

$$z(s,t) := (1+s)e^{2\pi it},\ 0 \le s,t \le 1$$

deforms continuously the circle $C_0(1)$ into the circle $C_0(2)$.

7.3. Deformation Invariance Theorem.

We first recall the definition of a *simply connected domain.*

Definition: Any domain D in the complex plane C possessing the property that every loop in D can be continuously deformed in D to a point is called simply connected. $\aleph$.

For example, any disk $D_a(r), r > 0$ is a simply connected domain.

Now we are in position to prove the *Deformation Invariance Theorem*.

Theorem 7.3. *Let* D *be a domain in* C *and suppose that* $f \in A(D)$. *If* γ_1, γ_2 *are continuously deformable into each other closed curves, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof:

Fix $s \in [0,1]$ and set $\gamma(s) := z(s,t), t \in [0,1]$. We shall show that the function $I(s) := \int_{\gamma(s)} f(z)dz$ equals a constant. Indeed,

$$\int_{\gamma(s)} f(z)dz = \int_{\gamma(s)} f(z(s,t))\frac{\partial z(s,t)}{\partial t}dt.$$

Look at the derivative of $I(s)$; we have

$$I'(s) = \int_{\gamma(s)} f(z(s,t))\frac{\partial z(s,t)}{\partial t}dt =$$

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$$= \int_{\gamma(s)} [\frac{\partial f(z(s,t))}{dt}\frac{\partial z(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t} + f(z(s,t))\frac{\partial^2 z(s,t)}{\partial s \partial t}]dt.$$

On the other hand,

$$\frac{\partial}{dt}(f(z(s,t))\frac{\partial z(s,t)}{\partial s}) = \frac{\partial f(z(s,t))}{dt}\frac{\partial z(s,t)}{\partial t}\frac{\partial z(s,t)}{\partial s} + f(z(s,t))\frac{\partial^2 z(s,t)}{\partial t \partial s}.$$

The theorem by Weierstrass about the independence of second order derivatives of the order of differentiation guarantees that

$$\frac{dI(s)}{ds} = \int_0^1 \frac{\partial}{dt}[f(z(s,t))\frac{\partial z(s,t)}{\partial s}]dt =$$

$$= f(z(s,1))\frac{\partial z(s,t)}{\partial s}(s,1) - f(z(s,0))\frac{\partial z(s,t)}{\partial s}(s,0).$$

As we know, the curves $\gamma(s)$ are closed which means that for every $s \in [0,1]$ $z(s,0) = z(s,1)$.

Thus

$$I(s) = \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Q.E.D.

Cauchy's integral theorem An easy consequence of Theorem 7.3. is the following, familiarly known as *Cauchy's integral theorem.*

Theorem 7.4.*If D is a simply connected domain, f* ∈ A(D) *and* Γ *is any loop in D, then*

$$\int_{\Gamma} f(z)dz = 0.$$

Proof: The proof follows immediately from the fact that each closed curve in *D* can be shrunk to a point. **Q.E.D.**

We conclude the following

Theorem 7.5. *Let* D *be a domain in* C *and* $f \in A(D)\bigcap C(\overline{D})$*. Set* $\partial D := \Gamma$*. Then*

$$\int_{\Gamma} f(z)dz = 0$$

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Proof: Without losing the generality, we may assume that all components of Γ are smooth curves. It D is simply connected, then we are done. Assume that D is double connected and let $\Gamma = \Gamma_1 \bigcup \Gamma_2$. The domain is positively orientated with respect to Γ; let Γ_1 be the positive component (clockwise) and Γ_2− the negative (counterclockwise) ($\Gamma = \Gamma_1 \bigcup (-\Gamma_2)$.) Without loosing the generality we suppose that Γ_1 and Γ_2 are continuously deformable into each other, and by Theorem 7.3.

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz. \qquad (1)$$

On the other hand

$$\int_{\Gamma} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{-\Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz = 0.$$

Joining (1), we arrive at the statement.

The Cauchy's integral theorem indicates the intimate relation between simply connectedness and existence of a continuous antiderivative.

Theorem 7.6. *Let* D *be simply connected in* C *and* $f \in A(D)$*.*

Then f possesses a continuous antiderivative and its contour integral does not depend on the path of integration.

The proof follows from Theorem 7.3.

7.4. Cauchy's integral formula

Theorem 7.7. *Let* D *be a domain in* C, Γ := ∂D *and* $f \in A(D)^{T}\overline{C}(D)$. *Then, for every point a* ∈ D *the representation*

$$f(a) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-a}dz \tag{2}$$

holds.

Proof:

Take r sufficiently small (e.g. $\overline{D_a(r)} \subset D$) and consider $\oint_{|z-a|=r}\frac{f(z)}{z-a}dz$.

(the circle is traversed once in the positive direction). We have

$$\frac{1}{2\pi i}\oint_{|z-a|=r}\frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{i\Theta})}{re^{i\Theta}}ire^{i\Theta}d\Theta$$

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Letting now $r \to 0$ we obtain that

$$\frac{1}{2\pi i}\oint_{|z-a|=r} f(z)dz = f(a)$$

To complete the proof, we apply Theorem 7.5. with respect to the function $\frac{f(z)}{z-a}$ and to the domain D \ $\overline{D_a(r)}$. **Q.E.D.**

As an application, we provide the *mean value theorem for harmonic functions.*

Theorem 7.7. *Let h be harmonic in the disk* $D_a(R)$, $R > 0$. *Then*

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a+Re^{i\Theta})d\Theta$$

Proof: : We recall that the real and the imaginary components of an analytic function are complex conjugate harmonic functions. Let $f \in A(D_a(R))$ be such that $h(z) := <f(z)$. Denote the imaginary component by $k(z)$.

$$f(f) = h(z) + ik(z),\ z \in K_a(R).$$

Using (2), we get

$$h(a) + ik(a) = \frac{1}{2\pi i}\int_{C_a(R)}\frac{h(\zeta)+ik(\zeta)}{\zeta - a}d\zeta.$$

Hence,

$$h(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{h(a+Re^{i\Theta})}{Re^{i\Theta}} iRe^{i\Theta} d\Theta$$
.

The statement follows after completing the needed cancellations.

Exercises:

1. Prove that

$$\int_{C_a(\rho)} \frac{dz}{(z-a)^m} = \begin{cases} 0, & m \neq 1 \\ 2\pi i, & m = 1 \end{cases}$$
♣

2. Prove that

$$\int_{C_0(\rho)} \frac{dz}{(z-a)} = \begin{cases} 0, & |a| > \rho \\ 2\pi i, & |a| < \rho \end{cases}$$
♣

3. Which of the following domains are simply connected?

a) $\{z, |\operatorname{Im} z| < 1\}$;

b) $\{z, 1 < |z| < 2\}$;

c) $\{z, |z| < 1\}$;

d) $\{z, |z| > 1\}$;

e) $\{z, |z| < 1\} \setminus \{z, 0 < \operatorname{Re} z < 1\}$. ♣

3. Calculate

$$\int_S \frac{1}{1+z^2} dz,$$

with S being the interval $[1, 1+i]$. ♣

4. Show that if $f(z)$ is of the form

$$f(z) = \sum_{k=0}^{n} \frac{A_k}{z^k} + g(z)$$
,

where $g(z)$ is analytic outside $C_0(1)$, then

$$\oint_{|z|=1} f(z)dz = 2\pi i A_1.$$

♣

(By definition, $\mathrm{H}_{|z|=1} := \mathrm{R}_{C_0(1)}, C_0(1)$ traversed once in positive direction.) ♣ 5. Let P be a polynomial of degree ≥ 2, such that all zeros lie in $D_0(R), R > 0$. Show that

$$\oint_{|z|=R} \frac{1}{P(z)} dz = 0.♣$$

Hint Apply Theorem 7.5. with respect to the annulus $\{z, R < |z| < R + r\}$ and then let r increase to infinity. ♣