Bessel Functions of the First Kind

Recall the *Bessel equation*

$$x^2 y'' + x y' + (x^2 - n^2) y = 0.$$

For a fixed value of n, this equation has two linearly independent solutions. One of these solutions, that can be obtained using Frobenius' method, is called a *Bessel function of the first kind*, and is denoted by $J_n(x)$. This solution is regular at $x = 0$. The second solution, that is singular at $x = 0$, is called a *Bessel function of the second kind*, and is denoted by $Y_n(x)$.

Generating Function for Integral Order

A *generating function* for a sequence $\{a_n\}$ is a power series

$$g(t) = \sum_n a_n t^n,$$

of which the terms of the sequence are the coefficients. Similarly, a generating function for a sequence of functions $\{f_n(x)\}$ is a power series

$$g(x, t) = \sum_n f_n(x) t^n,$$

whose coefficients are now functions of x.

The generating function for the sequence of Bessel functions of the first kind, of integer order, is

$$g(x, t) = e^{(x/2)(t - 1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n.$$

To obtain an expression for $J_n(x)$, we use the Maclaurin series for e^x to obtain

$$
\begin{aligned}
g(x, t) &= e^{xt/2} e^{-x/(2t)} \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{x}{2}\right)^r t^r \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{x}{2}\right)^s t^{-s} \\
&= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s}{r! s!} \left(\frac{x}{2}\right)^{r+s} t^{r-s} \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{s=\max\{0,-n\}}^{\infty} \frac{(-1)^s}{(n+s)! s!} \left(\frac{x}{2}\right)^{n+2s} \right] t^n
\end{aligned}
$$

It follows that for $n \geq 0$,

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)!s!} \left(\frac{x}{2}\right)^{n+2s},$$

while for $n \geq 0$, we have

$$J_{-n}(x) = \sum_{s=n}^{\infty} \frac{(-1)^s}{(s-n)!s!} \left(\frac{x}{2}\right)^{-n+2s}.$$

Using an index shift, we obtain

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{(n+s)!s!} \left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x).$$

For noninteger n, rather than using the generating function, we can use Frobenius' method to obtain

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{\nu+2s},$$

where Γ is the *Gamma function*. As $\Gamma(s) = (s-1)!$ for s a positive integer, this formula is consistent with the one for integer order.

Recurrence Relations

Using the generating function $g(x,t)$, we can obtain some useful *recurrence relations* involving Bessel functions of the first kind. Differentiating $g(x,t)$ with respect to x and t yields

$$\frac{\partial g(x,t)}{\partial x} = \frac{1}{2}\left(t - \frac{1}{t}\right)e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n'(x)t^n,$$

$$\frac{\partial g(x,t)}{\partial t} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right)e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}.$$

Replacing the formula for $g(x,t)$ with its power series expansion in the above equations yields

$$\sum_{n=-\infty}^{\infty} J_n'(x)t^n = \frac{1}{2}\left(t - \frac{1}{t}\right)\sum_{n=-\infty}^{\infty} J_n(x)t^n = \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)(t^{n+1} - t^{n-1})$$

$$\sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)t^n = \frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)(t^n + t^{n-2}).$$

Using index shifts, we obtain

$$\sum_{n=-\infty}^{\infty} J_n'(x)t^n = \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)(t^{n+1} - t^{n-1})$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^{n+1} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^{n-1}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n-1}(x)t^n - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n+1}(x)t^n$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} [J_{n-1}(x) - J_{n+1}(x)]t^n,$$

$$\sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1} = \frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)(t^n + t^{n-2})$$

$$= \frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^n + \frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^{n-2}$$

$$= \frac{x}{2}\sum_{n=-\infty}^{\infty} J_{n-1}(x)t^{n-1} + \frac{x}{2}\sum_{n=-\infty}^{\infty} J_{n+1}(x)t^{n-1}$$

$$= \frac{x}{2}\sum_{n=-\infty}^{\infty} [J_{n-1}(x) + J_{n+1}(x)]t^{n-1}.$$

Matching power series coefficients, we obtain the recurrence relations

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$
$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

From these recurrence relations, we can obtain the formulas

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x),$$
$$\frac{d}{dx}[x^{-n} J_n(x)] = x^{-n} J_{n+1}(x),$$
$$J_n(x) = \pm J_{n\pm1}'(x) + \frac{n\pm1}{x} J_{n\pm1}(x)$$

Integral Representation

It is quite useful to have an integral representation of Bessel functions. From complex analysis, the *residue theorem* states that if a function $f(z)$ defined on the complex plane has the *Laurent series* representation

$$\infty f(z) = {}^X a_n(z -$$

$$z_0)^n, \; n=-\infty$$

and if C is any positively oriented simple closed curve in the complex plane containing z_0, then

$$I f(z)dz = 2\pi i a_{-1}.$$
$$C$$

Therefore, if we apply this theorem to the generating function for the Bessel functions of integer order, we obtain

$$\oint_C \frac{e^{(x/2)(t-1/t)}}{t^{n+1}}\, dt = \oint \sum_{m=-\infty}^{\infty} J_m(x) t^{m-n-1}\, dt = 2\pi i J_n(x)$$

,

where C is any positively oriented simple closed curve in the complex plane containing the origin.

Now, suppose that we choose C to be the the unit circle, and we use the substitution $t = e^{i\theta}$, for which $dt = ie^{i\theta}\, d\theta$. Then we have

$$2\pi i J_n(x) = i \int_0^{2\pi} \frac{e^{(x/2)(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}} e^{i\theta}\, d\theta = i \int_0^{2\pi} e^{i(x\sin\theta - n\theta)}\, d\theta$$

which yields

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x\sin\theta - n\theta)} = \frac{1}{2\pi} \int_0^{2\pi} \cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)\, d\theta.$$

Taking the real and imaginary parts of both sides of the equation, we obtain

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(x\sin\theta - n\theta)\, d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta,$$

where the second integral is obtained via symmetry, and

$$\int_0^{2\pi} \sin(x\sin\theta - n\theta)\, d\theta = 0$$

.

A special case of particular interest is

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta)\, d\theta.$$

Also, it is worth noting that although these integral representations were derived only for Bessel functions of integer order, the relation on which they are based also applies to Bessel functions of noninteger order. Specifically,

$$J_\nu(x) = \frac{1}{2\pi i} \oint_C \frac{e^{(x/2)(t-1/t)}}{t^{\nu+1}}\, dt,$$

where C is a contour in the complex plane that encircles the origin $t = 0$.

Bessel Functions of Noninteger Order

Bessel functions of noninteger order satisfy the same recurrence relations as those of integer order, as can be proven using the power series representation given earlier. However, one key difference between Bessel functions of integer and noninteger order is that if ν is not an integer, then J_ν and $J_{-\nu}$ are linearly independent solutions of the Bessel equation of order ν, which means that the relation $J_{-\nu} = (-1)^\nu J_\nu$ does not hold. However, if ν is an integer, then we must use some technique other than Frobenius' method, such as reduction of order, to obtain a second linearly independent solution of the Bessel equation of order ν.

The Bessel Equation

We have seen that Bessel functions of the first kind, which are solutions of the Bessel equation, satisfy the recurrence relations

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x), \quad J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x)$$

Now, suppose that a sequence of functions $\{Z_\nu(x)\}$, but not necessarily Bessel functions, all satisfy the above recurrence relations. Then, we have

$$
\begin{aligned}
x^2 Z_\nu'' + x Z_\nu' - \nu^2 Z_\nu &= x^2 \left(\frac{Z_{\nu-1} - Z_{\nu+1}}{2}\right)' + x\frac{Z_{\nu-1} - Z_{\nu+1}}{2} - \nu^2 Z_\nu \\
&= \frac{x^2}{2}\left[Z_{\nu-1}' + \frac{1}{x}Z_{\nu-1} - Z_{\nu+1}' - \frac{1}{x}Z_{\nu+1}\right] - \nu^2 \frac{x}{2\nu}[Z_{\nu-1} + Z_{\nu+1}] \\
&= \frac{x^2}{2}\left[Z_{\nu-1}' - \frac{\nu-1}{x}Z_{\nu-1} - Z_{\nu+1}' - \frac{\nu+1}{x}Z_{\nu+1}\right] \\
&= \frac{x^2}{2}[-2Z_\nu] \\
&= -x^2 Z_\nu.
\end{aligned}
$$

In other words,

$$x^2 Z_\nu'' + x Z_\nu' + (x^2 - \nu^2)Z_\nu = 0,$$

meaning that Z_ν is a solution of the Bessel equation.

Now, suppose that $Z_\nu(x)$ is replaced by $Z_\nu(kx)$ for some constant k. Proceeding as before, and using the Chain Rule, we obtain

$$
\begin{aligned}
x^2 Z_\nu'' + x Z_\nu' - \nu^2 Z_\nu &= k^2 x^2 \left(\frac{Z_{\nu-1} - Z_{\nu+1}}{2}\right)' + kx\frac{Z_{\nu-1} - Z_{\nu+1}}{2} - \nu^2 Z_\nu \\
&= \frac{k^2 x^2}{2}\left[Z_{\nu-1}' + \frac{1}{kx}Z_{\nu-1} - Z_{\nu+1}' - \frac{1}{kx}Z_{\nu+1}\right] - \nu^2 \frac{kx}{2\nu}[Z_{\nu-1} + Z_{\nu+1}] \\
&= \frac{k^2 x^2}{2}\left[Z_{\nu-1}' - \frac{\nu-1}{kx}Z_{\nu-1} - Z_{\nu+1}' - \frac{\nu+1}{kx}Z_{\nu+1}\right] \\
&= \frac{k^2 x^2}{2}[-2Z_\nu] \\
&= -k^2 x^2 Z_\nu,
\end{aligned}
$$

where all Z_ν, $Z_{\nu-1}$ and $Z_{\nu+1}$, and their derivatives, are evaluated at kx. It follows that

$$x^2 Z_\nu''(kx) + x Z_\nu'(kx) + (k^2 x^2 - \nu^2)Z_\nu(kx) = 0.$$

This slightly modified form of the Bessel equation will arise when solving partial differential equations (PDE) using *separation of variables*.