ENTRI

Bessel Functions of the Second Kind

When solving the Bessel equation of integer order, Frobenius' method only produces one linearly independent solution. A second solution may be found using reduction of order, but it is not of the same form as a Bessel function of the first kind. Therefore, we refer to it as a *Bessel function of the second kind*, which is also known as a *Neumann function*.

Definition and Series Form

The Neumann function of order ν is defined as follows:

$$Y_\nu(x) = \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}.$$

This function is clearly a solution of the Bessel equation, as it is a linear combination of solutions. However, if ν is an integer, then $Y_\nu(x)$, as defined, is the indeterminate form $0/0$. Therefore, we need to use l'Hospital's Rule to determine whether the limit as ν approaches an integer n is nonzero, so that we can obtain a meaningful solution. Applying l'Hospital's Rule yields

$$\begin{aligned} Y_n(x) &= \lim_{\nu\to n} \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \\ &= \lim_{\nu\to n} \frac{-\pi\sin\nu\pi J_\nu(x) + \cos\nu\pi \frac{d}{d\nu}J_\nu(x) - \frac{d}{d\nu}J_{-\nu}(x)}{\pi\cos\nu\pi} \\ &= \lim_{\nu\to n} \frac{1}{\pi}\left[\frac{d}{d\nu}J_\nu(x) - (-1)^n\frac{d}{d\nu}J_{-\nu}(x)\right]. \end{aligned}$$

Using the series definition of $J_\nu(x)$,

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},$$

as well as the *digamma function*

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

we obtain

$$\begin{aligned} Y_n(x) &= \frac{1}{\pi}\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\ln\left(\frac{x}{2}\right) + (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k-n)!}\left(\frac{x}{2}\right)^{2k-n}\ln\left(\frac{x}{2}\right)\right] - \\ &\quad \frac{1}{\pi}\left[\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma'(k+n+1)}{k!(k+n)!^2}\left(\frac{x}{2}\right)^{2k+n} + (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma'(k-n+1)}{k!(k-n)!^2}\left(\frac{x}{2}\right)^{2k-n}\right] \end{aligned}$$

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\begin{aligned}
&= \frac{1}{\pi}\left[J_n(x)\ln\left(\frac{x}{2}\right) + (-1)^n J_{-n}(x)\ln\left(\frac{x}{2}\right)\right] - \\
&\quad \frac{1}{\pi}\left[\sum_{k=0}^{\infty}\frac{(-1)^k\psi(k+n+1)}{k!(k+n)!^2}\left(\frac{x}{2}\right)^{2k+n} + (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k\psi(k-n+1)}{k!(k-n)!^2}\left(\frac{x}{2}\right)^{2k-n}\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \\
&\quad \frac{1}{\pi}\left[\sum_{k=0}^{\infty}\frac{(-1)^k\psi(k+n+1)}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} + (-1)^n\sum_{k=-n}^{\infty}\frac{(-1)^{k+n}\psi(k+1)}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{(-1)^n}{\pi}\sum_{k=-n}^{-1}\frac{(-1)^{k+n}\psi(k+1)}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{(-1)^n}{\pi}\sum_{k=0}^{n-1}\frac{(-1)^k\psi(k-n+1)}{k!(k-n)!}\left(\frac{x}{2}\right)^{2k-n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{(-1)^n}{\pi}\sum_{k=0}^{n-1}\frac{(-1)^k(-1)^{n-k-2}(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right]
\end{aligned}
$$

where we have used the result

$$\lim_{z\to -n}\frac{\psi(z)}{\Gamma(z)} = (-1)^{n-1}n!$$

We can see from the above expression for $Y_n(x)$ that it is indeed linearly independent of $J_n(x)$, so that we have two linearly independent solutions of the Bessel equation for integer order n. Also, unlike $J_n(x)$, $Y_n(x)$ is singular at $x = 0$.

For $n = 0$, we have

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi} J_0(x) \ln\left(\frac{x}{2}\right) - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} \left(\frac{x}{2}\right)^{2k} \psi(k+1) \\ &= \frac{2}{\pi} J_0(x) \ln\left(\frac{x}{2}\right) - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} \left(\frac{x}{2}\right)^{2k} [-\gamma + H_k] \\ &= \frac{2}{\pi} J_0(x) \left[\gamma + \ln\left(\frac{x}{2}\right)\right] - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} \left(\frac{x}{2}\right)^{2k} H_k \end{aligned}$$

where γ is the Euler-Mascheroni constant. Here we have used the relation

$$\psi(n+1) = -\gamma + \sum_{m=1}^{n} \frac{1}{m}$$

when n is a positive integer.

Integral Representations

Bessel functions of the second kind also have integral representations. We have

$$\begin{aligned} Y_0(x) &= -\frac{2}{\pi} \int_0^{\infty} \cos(x \cosh t)\, dt = -\frac{2}{\pi} \int_1^{\infty} \frac{\cos(xt)}{(t^2-1)^{1/2}}\, dt, \quad x > 0, \\ Y_n(x) &= \frac{1}{\pi} \int_0^{\pi} \sin(x \sin t - nt)\, dt - \frac{1}{\pi} \int_0^{\infty} [e^{nt} + (-1)^n e^{-nt}] e^{-x \sinh t}\, dt. \end{aligned}$$

Recurrence Relations

Bessel functions of the second kind, being solutions of the Bessel equation, satisfy the same recurrence relations as the Bessel functions of the first kind. Specifically,

$$\begin{aligned} Y_{\nu-1}(x) - Y_{\nu+1}(x) &= 2Y_\nu(x), \\ Y_{\nu-1}(x) + Y_{\nu+1}(x) &= \frac{2\nu}{x} Y_\nu(x). \end{aligned}$$

We also have the relation

$$Y_{-n}(x) = (-1)^n Y_n(x),$$

when n is an integer.

Wronskian Formulas

If $y_1(x)$ and $y_2(x)$ are solutions of a self-adjoint ODE of the form $p(x)y^{00} + q(x)y^0 + r(x)y = 0$, for which $q(x) = p^0(x)$, we can use Abel's Theorem to obtain the Wronskian

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = Ce^{-\int \frac{p'(x)}{p(x)}\, dx} = \frac{C}{p(x)},$$

where C is a constant.

By writing the Bessel equation in the form

$$xy^{00} + y^0 + (x - \nu^2/x)y = 0,$$

so that it is self-adjoint, we obtain, for non-integer ν,

$$J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x) = \frac{A_\nu}{x},$$

where A_ν is a constant that depends only on ν, not x.

This constant can be determined by considering any convenient value of x, such as x = 0. Examining the leading terms of the series representations of the Bessel functions, which yield approximations for small x,

$$\begin{aligned} J_\nu(x) &\approx \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \\ J'_\nu(x) &\approx \frac{\nu}{2\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu-1}, \\ J_{-\nu}(x) &\approx \frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu}, \\ J'_{-\nu}(x) &\approx \frac{-\nu}{2\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}, \end{aligned}$$

we obtain

$$W(J_\nu, J_{-\nu}) = -\frac{2\nu}{x\Gamma(1+\nu)\Gamma(1-\nu)} = -\frac{2\sin\nu\pi}{\pi x}.$$

We conclude that

$$A_\nu = -\frac{2\sin\nu\pi}{x}.$$

When ν is an integer, we obtain A_ν = 0, and therefore the Wronskian is zero. This is expected, since J_n and J_{-n} are linearly dependent when n is an integer.

Using recurrence relations, we obtain the following similar formulas:

$$\begin{aligned} J_\nu J_{-\nu+1} + J_{-\nu}J_{\nu-1} &= \frac{2\sin\nu\pi}{\pi x}, \\ J_\nu J_{-\nu-1} + J_{-\nu}J_{\nu+1} &= -\frac{2\sin\nu\pi}{\pi x} \\ J_\nu Y'_\nu - J'_\nu Y_\nu &= \frac{2}{\pi x}, \\ J_\nu Y_{\nu+1} - J_{\nu+1}Y_\nu &= -\frac{2}{\pi x}. \end{aligned}$$

Uses of Neumann Functions

Besides completing a set of linearly independent solutions of the Bessel equation, Neumann functions $Y_\nu(x)$ also are useful for physical problems in which there is no requirement of regularity at x = 0, such as when modeling electromagnetic waves in coaxial cables, or in quantum mechanical scattering theory.